Example 1.6 Evaluate $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2}$.

Solution: $\lim\limits_{x \to 2} \dfrac{x^2-4}{x-2} = \lim\limits_{x \to 2} \dfrac{(x+2)(x-2)}{x-2}$

$= \lim\limits_{x \to 2} x+2 = 2+2 = 4.$

Example 1.7 Evaluate $\lim\limits_{x \to \infty} \dfrac{3x^2-2x-3}{x^2+4x+4}$.

Solution

$$\lim_{x \to \infty} \frac{3x^2-2x-3}{x^2+4x+4} = \lim_{x \to \infty} \frac{(3x^2-2x-3)\frac{1}{x^2}}{(x^2+4x+4)\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{3-\frac{2}{x}-\frac{3}{x^2}}{1+\frac{4}{x}+\frac{4}{x^2}} = \cancel{\lim_{x\to\infty}} \frac{3-0-0}{1+0+0} = \frac{3}{1} = 3.$$

Example 1.8 Evaluate $\lim\limits_{x \to \infty} \left(\sqrt{x^2+2} - x\right)$

Solution: $\lim\limits_{x \to \infty} \left(\sqrt{x^2+2} - x\right) = \lim\limits_{x \to \infty} \dfrac{\left(\sqrt{x^2+2} - x\right)\left(\sqrt{x^2+2}+x\right)}{\sqrt{x^2+2}+x}$

$$= \lim_{x \to \infty} \frac{x^2+2-x^2}{\sqrt{x^2+2}+x} = \lim_{x \to \infty} \frac{2}{\sqrt{x^2+2}+x}$$

$$\leq \lim_{x \to \infty} \frac{2}{\sqrt{x^2}+x} = \lim_{x \to \infty} \frac{2}{2x} = \lim_{x \to \infty} \frac{1}{x} = 0.$$

Example 1.9 Evaluate $\lim_{x\to 0} x^2\sin(\frac{1}{x})$.

Solution: Since $|x^2\sin(\frac{1}{x})| = |x^2||\sin(\frac{1}{x})| \le |x^2|\cdot 1 = |x^2|$

then

$$\lim_{x\to 0} |x^2\sin(\tfrac{1}{x})| \le \lim_{x\to 0} |x^2| = 0.$$

So $\lim_{x\to 0} |x^2\sin\frac{1}{x}| = 0$, and so $\lim_{x\to 0} x^2\sin(\frac{1}{x}) = 0$.

Example 1.10 Evaluate $\lim_{x\to 0} x\sin(\frac{1}{x})$.

Solution Since $|x\sin(\frac{1}{x})| = |x|\cdot|\sin(\frac{1}{x})| \le |x|\cdot 1 = |x|$

then

$$\lim_{x\to 0} |x\sin(\tfrac{1}{x})| \le \lim_{x\to 0} |x| = 0$$

So $\lim_{x\to 0} |x\sin(\frac{1}{x})| = 0$ and so $\lim_{x\to 0} x\sin(\frac{1}{x}) = 0$.

Example 1.11 Let $f(x) = \begin{cases} 2x, & \text{if } x \ne 1, \\ 4, & \text{if } x = 1. \end{cases}$

Is f continuous at $x=1$?

Solution $\lim_{x\to 1} f(x) = \lim_{x\to 1} 2x = 2\cdot 1 = 2$

and $f(1) = 4$. So $\lim_{x\to 1} f(x) \ne f(1)$

So f is <u>not</u> continuous at $x=1$.

Example 1.12 Let $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x \neq 2 \\ 4, & \text{if } x = 2 \end{cases}$

Is f continuous at $x=2$?

Solution:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x\to 2} x+2$$

$$= 2+2 = 4, \text{ and}$$

$f(2) = 4$.

So $\lim_{x\to 2} f(x) = f(2)$. So f is continuous at $x=2$.

Example 1.13 Is $f(x) = \sqrt{x}$ continuous in its domain?

Solution Probably you mean $\begin{array}{rl} f: \mathbb{R}_{\geq 0} & \to \mathbb{R}_{\geq 0} \\ x & \mapsto \sqrt{x} \end{array}$

so that the domain (source) is $\mathbb{R}_{\geq 0}$.

~~The~~ If $a \in \mathbb{R}_{\geq 0}$

$$\lim_{x\to a} \sqrt{x} = \sqrt{a} \quad \text{and} \quad f(a) = \sqrt{a}.$$

So, if $a \in \mathbb{R}_{\geq 0}$ then $\lim_{x\to a} f(x) = f(a)$

So, if $a \in \mathbb{R}_{\geq 0}$ then f is continuous at $x=a$.