Calculus 2 Lecture 22

16.09.2019
A. Ram

First order differential equations

$$\frac{dy}{dx} + Q(x)y = R(x)$$

Second order differential equations

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$$

Solution will be

$$y = c_1 E_1 + c_2 E_2,$$ where c_1 and c_2 are constants

and E_1 and E_2 are linearly independent functions of x.

Initial value problem Use

$$y(t_0) = p_0 \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x=t_0} = V_0$$

to determine c_1 and c_2.

Boundary value problem Use

$$y(t_0) = p_0 \quad \text{and} \quad y(t_1) = q_1$$

to determine c_1 and c_2.

Homogeneous 2^{nd} order differential equations

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) = 0.$$

Constant coefficient homogeneous 2^{nd} order differential equations:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

The <u>discriminant</u> is $\Delta = b^2 - 4ac$.

(a) If $\Delta \neq 0$ then the solutions are

$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$, where c_1 and c_2 are constants

and $aD^2 + bD + c = a(D-\lambda_1)(D-\lambda_2)$

(b) If $\Delta = 0$ then the solutions are

$y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$; where c_1 and c_2 are constants

and $aD^2 + bD + c = a(D-\lambda)^2$.

<u>Example 6.3</u> Solve $\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0$.

<u>Solution</u>: $D^2 + 7D + 12 = (D+3)(D+4)$.

So $y = c_1 e^{-3x} + c_2 e^{-4x}$, where c_1 and c_2 are constants.

<u>Check</u>: Let $y = c_1 e^{-3x} + c_2 e^{-4x}$. Then

$\frac{dy}{dx} = -3c_1 e^{-3x} - 4c_2 e^{-4x}$ and

$$\frac{d^2y}{dx^2} = 9c_1e^{-3x} + 16c_2e^{-4x} \quad \text{and}$$

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 9c_1e^{-3x} + 16c_2e^{-4x}$$
$$+ (-21)c_1e^{-3x} - 28c_2e^{-4x}$$
$$+ 12c_1e^{-3x} + 12c_2e^{-4x}$$
$$= 0c_1e^{-3x} + 0c_2e^{-4x} = 0. //$$

<u>Example 6.4</u> Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$.

<u>Solution</u> Since $D^2 + 2D + 1 = (D+1)^2$ the solutions are

$y = c_1e^{-x} + c_2xe^{-x}$, where c_1 and c_2 are constants.

Check: Let $y = c_1e^{-x} + c_2xe^{-x}$. Then

$$\frac{dy}{dx} = -c_1e^{-x} + c_2(e^{-x} + x(-1)e^{-x})$$
$$= (-c_1 + c_2)e^{-x} - c_2xe^{-x}, \quad \text{and}$$

$$\frac{d^2y}{dx^2} = -(-c_1+c_2)e^{-x} - c_2(e^{-x} - xe^{-x})$$
$$= (c_1 - c_2)e^{-x} - c_2e^{-x} + c_2xe^{-x}$$
$$= (c_1 - 2c_2)e^{-x} + c_2xe^{-x}.$$

Then

$$\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=(c_1-2c_2)e^{-x}+c_2xe^{-x}$$
$$+2(c_2-c_1)e^{-x}-2c_2xe^{-x}$$
$$+c_1e^{-x}+c_2xe^{-x}$$
$$=0e^{-x}+0xe^{-x}=0.$$

<u>Example 6.5</u> Solve $\frac{d^2y}{dx^2}-4\frac{dy}{dx}+13y=0$ with $y(0)=-1$ and $y'(0)=2$.

<u>Solution:</u> $D^2-4D+13=(D-(2+3i))(D-(2-3i))$ since the quadratic formula gives

$$\frac{-(-4)\pm\sqrt{(-4)^2-4\cdot 13}}{2}=\frac{4\pm\sqrt{4(-9)}}{2}=2\pm 3i$$

So

$$y=c_1e^{(2+3i)x}+c_2e^{(2-3i)x},$$ where c_1 and c_2 are constants.

Since

$$-1=y(0)=c_1e^0+c_2e^0=c_1+c_2 \quad \text{and}$$

$$2=\left.\frac{dy}{dx}\right|_{x=0}=\left.\left((2+3i)c_1e^{(2+3i)x}+(2-3i)c_2e^{(2-3i)x}\right)\right|_{x=0}$$
$$=(2+3i)c_1e^0+(2-3i)c_2e^0$$
$$=(2+3i)c_1+(2-3i)c_2=2(c_1+c_2)+3i(c_1-c_2).$$

then
$$2 = 2(-1) + 3i(c_1 - c_2) \text{ and}$$

$$c_1 + c_2 = -1 \text{ and } c_1 - c_2 = \frac{4}{3i} = -\frac{4}{3}i$$

So $2c_1 = (c_1+c_2) + (c_1 - c_2) = -1 - \frac{4}{3}i$ and $c_1 = \frac{-1}{2} - \frac{2}{3}i$

$2c_2 = (c_1+c_2) - (c_1 - c_2) = -1 + \frac{4}{3}i$ and $c_2 = \frac{-1}{2} + \frac{2}{3}i$

<u>Note:</u> $c_2 = \overline{c_1}$.

Write c_1 in polar form: $c_1 = \frac{-1}{2} - \frac{2}{3}i = re^{i\theta}$

where $r = \sqrt{\left(\frac{-1}{2}\right)^2 + \left(\frac{-2}{3}\right)^2} = \sqrt{\frac{1}{4} + \frac{4}{9}} = \sqrt{\frac{25}{36}} = \frac{5}{6}$ and

$$\theta = \arctan\left(\frac{-2/3}{-\frac{1}{2}}\right) = \arctan\left(\frac{4}{3}\right).$$

Then
$$\begin{aligned}
y &= c_1 e^{(2+3i)x} + c_2 e^{(2-3i)x} \\
&= e^2\left(c_1 e^{3ix} + c_2 e^{-3ix}\right) \\
&= e^2\left(re^{i\theta} e^{3ix} + re^{-i\theta} e^{-3ix}\right) \\
&= re^2\left(e^{i(3x+\theta)} + e^{-i(3x+\theta)}\right) \\
&= 2re^2 \tfrac{1}{2}\left(e^{i(3x+\theta)} + e^{-i(3x+\theta)}\right) \\
&= 2re^2 \cos(3x+\theta) \\
&= 2\cdot\tfrac{5}{6} e^2 \cos(3x+\theta) \\
&= \tfrac{5}{3} e^2 \cos\left(3x + \arctan\left(\tfrac{4}{3}\right)\right)
\end{aligned}$$