MAST10005 Slide
A. Ram 23.07.2019

Complex numbers

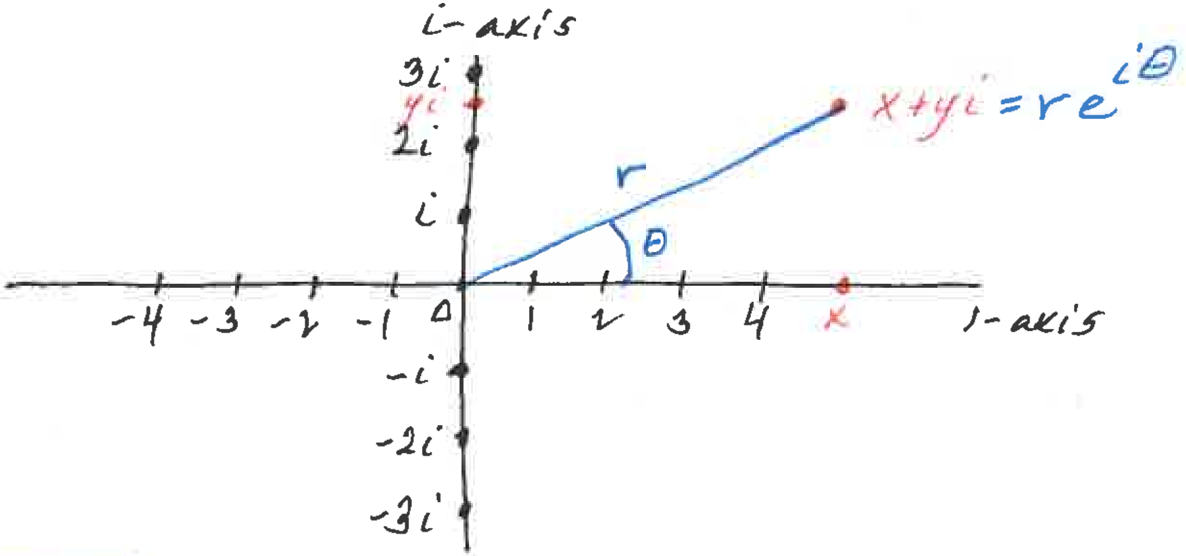
$\mathbb{C} = \{a+bi \mid a,b \in \mathbb{R}\}$ with $i^2=-1$.

$(a+bi)+(c+di) = (a+c)+(b+d)i$ (addition)

$c(a+bi) = ca + cbi$ (scalar multiplication)

$(a+bi)(c+di) = ac+adi+bci+bdi^2$

$= (ac-bd)+(ad+bc)i$ (multiplication)



Graphing of real and complex numbers.

$\overline{a+bi} = a-bi$ (conjugation)

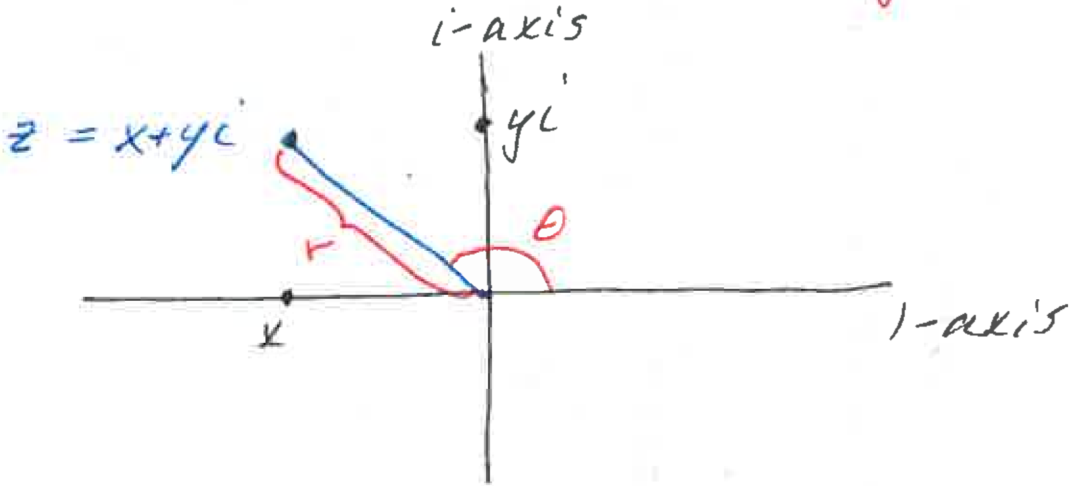
MAST10005

A. Ram 23.07.2019

Complex numbers

$$z = \underbrace{x}_{\mathrm{Re}(z)} + \underbrace{y}_{\mathrm{Im}(z)} i = \underbrace{r}_{|z|} e^{i\overbrace{\theta}^{\arg(z)}} \quad \text{with} \quad x \in \mathbb{R},\ y \in \mathbb{R}$$

$$r \in \mathbb{R}_{>0},\quad \theta \in \mathbb{R}_{[0, 2\pi)}$$



$|z|$ is the <u>modulus</u> of z.

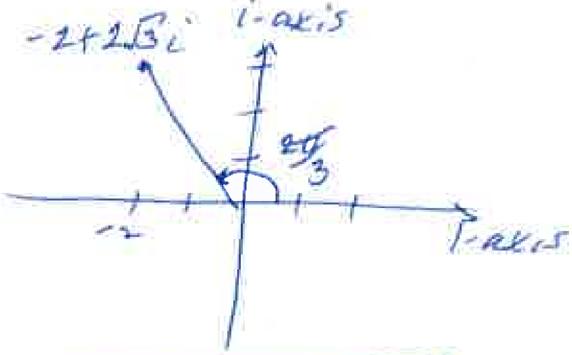
$re^{i\theta} = r\cos\theta + r\sin\theta\, i$, since

$x = r\cos\theta,\quad y = r\sin\theta$

$r = \sqrt{x^2+y^2},\quad \theta = \arctan(y/x)$

Example 1.50 Express $z = -2 + 2\sqrt{3}\,i$ in polar form.

Solution



$$|-2+2\sqrt{3}\,i| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{4+4\cdot 3} = 2\sqrt{4} = 2\cdot 2$$

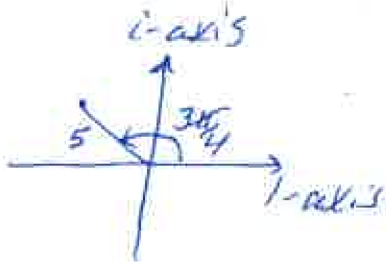
$= 4$ and

$$\text{Arg}(-2+2\sqrt{3}\,i) = \arctan\left(\frac{2\sqrt{3}}{-2}\right) = \frac{2\pi}{3} \text{ so that}$$

$$z = 4e^{i\pi/3}$$

Example 1.51 Express $z = 5e^{i\,3\pi/4}$ in Cartesian form.

Solution



$$z = 5e^{i\frac{3\pi}{4}} = 5\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$

$$= +5\left(\frac{-1}{\sqrt{2}} + i\,\frac{1}{\sqrt{2}}\right) = -\frac{5\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}\,i$$

Example 1.52 Using $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ prove that $e^{-i\theta} = \overline{e^{i\theta}}$.

Solution

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos(\theta) - i\sin\theta$$

$$= \overline{\cos\theta + i\sin\theta}$$

$$= \overline{e^{i\theta}}$$

Example 1.53 Use

$$\cos(\theta_1+\theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \quad \text{and}$$

$$\sin(\theta_1+\theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$$

to prove $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$

Solution

$$e^{i\theta_1}e^{i\theta_2} = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= \cos\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2$$

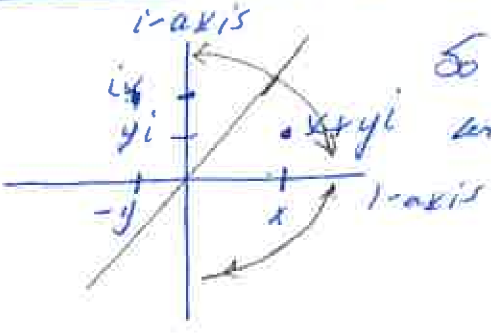
$$= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)$$

$$= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)}$$

//.

Example 1.56 Describe geometrically what happens when a complex number z is multiplied by i.

Solution If $z = x+iy$ then $iz = ix + i^2 y = -y + ix$



So x changes to $-y$
and y changes to x
So z gets "rotated by $\frac{\pi}{2}$"

<u>Example 1.56</u> Describe geometrically what happens when a complex number z is multiplied by i.

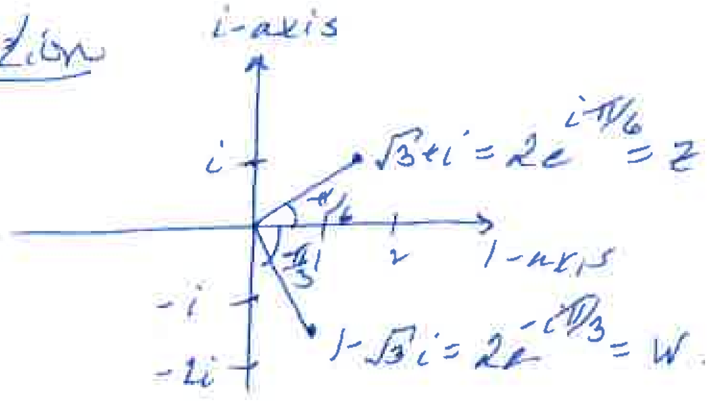
<u>Solution</u> If $z = re^{i\theta}$ then

$$iz = e^{i\frac{\pi}{2}} \cdot re^{i\theta} = re^{i(\theta + \frac{\pi}{2})}$$

So z gets "rotated by $\frac{\pi}{2}$".

<u>Example 1.57</u> Let $z = \sqrt{3} + i$ and $w = 1 - \sqrt{3}\,i$. Use polar form to find $\frac{1}{z}$ and $\frac{z}{w}$.

<u>Solution</u>



So

$$\frac{1}{z} = \frac{1}{2e^{i\pi/6}} = \frac{1}{2}e^{-i\frac{\pi}{6}} = \frac{1}{2}\left(\cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)\right)$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \frac{\sqrt{3}}{4} - \frac{1}{4}i, \text{ and}$$

$$\frac{z}{w} = \frac{2e^{i\pi/6}}{2e^{-i\pi/3}} = e^{-i\frac{\pi}{6}} = \cos\left(\frac{-\pi}{6}\right) + i\sin\left(\frac{-\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{2} - \frac{1}{2}i.$$