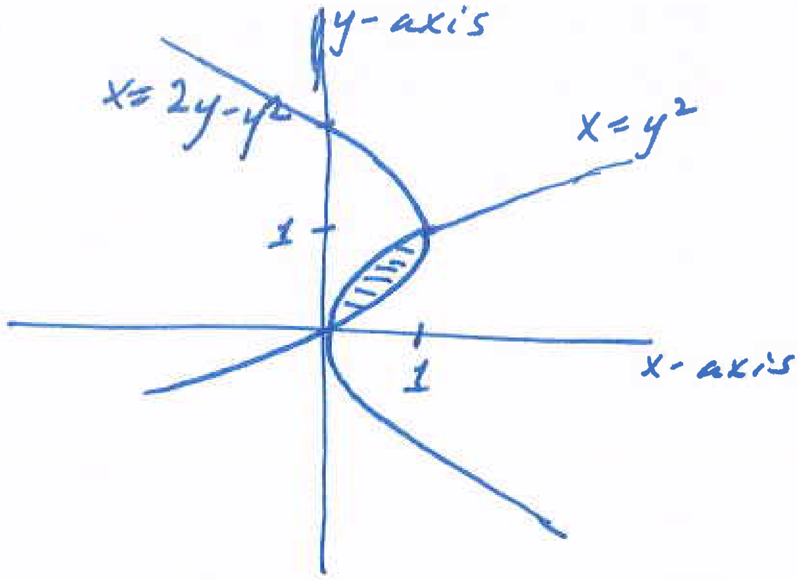
Q6 1

(6)



Point of intersection of $x = y^2$ and $x = 2y - y^2$ is at $y^2 = 2y - y^2$. So $2y^2 = 2y$. So $2y^2 - 2y = 0$

So $2y(y-1) = 0$.

So points of intersection are at $y = 0$ and $y = 1$.

The moment of inertia about the x-axis is

$$\iint_D y^2 \rho \, dx\, dy = \int_{y=0}^{y=1} \int_{x=y^2}^{x=2y-y^2} y^2 (y+1)\, dx\, dy$$

$$= \int_{y=0}^{y=1} y^2 (y+1) x \Big|_{x=y^2}^{x=2y-y^2} dy$$

$$= \int_{y=0}^{y=1} y^2 (y+1)(2y - y^2 - y^2)\, dy$$

$$= \int_{y=0}^{y=1} y^2 (y+1)\, 2y(1-y)\, dy = \int_{y=0}^{y=1} 2y^3 (1-y^2)\, dy$$

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$$= \frac{2}{4}y^4 - \frac{2}{6}y^6 \Big|_{y=0}^{y=1} = \frac{2}{4} - \frac{2}{6} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

(7) (a) Let $f = xy^2z^3 + 2z$. Then

$$\vec{\nabla} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$= (y^2z^3,\ 2xyz^3,\ 3xy^2z^2 + 2)$$

$$= y^2z^3\,\hat{i} + 2xyz^3\,\hat{j} + (3xy^2z^2 + 2)\,\hat{k} = \vec{F}.$$

(b) Since $\vec{F}$ is conservative, $\vec{\nabla} f = \vec{F}$, then the work done by $\vec{F}$ along C

$$\int_C \vec{F} \cdot d\vec{s} = f(\text{endpoint}) - f(\text{initial point}).$$

The points on the curve satisfy

$z = 2x + 3y$ and the initial point

$x^2 + y^2 = 12$ and endpoint are at $y = 0$.

When $y = 0$ then $x^2 = 12$ and $x = 2\sqrt{3}$

and $z = 2 \cdot 2\sqrt{3} + 3 \cdot 0 = 4\sqrt{3}$ is the ~~end~~ initial point

and $x = -2\sqrt{3}$ and $z = -4\sqrt{3}$ is the ~~initial~~ end point.

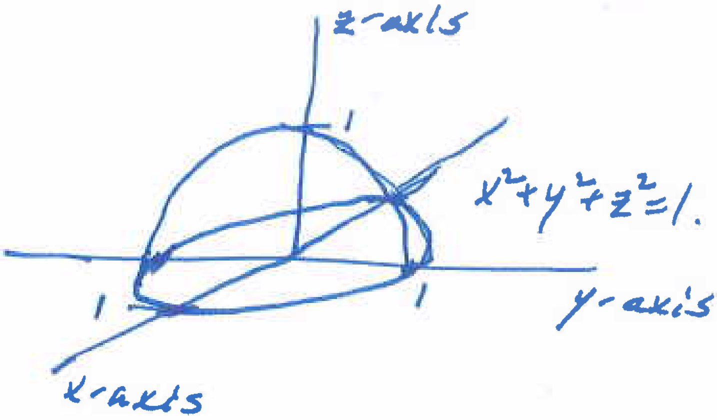
So

$$\text{Work} = -f(2\sqrt{3}, 0, 4\sqrt{3}) + f(-2\sqrt{3}, 0, -4\sqrt{3})$$

$$= -(2\sqrt{3} \cdot 0^2 \cdot (4\sqrt{3})^3 + 2 \cdot 4\sqrt{3}) + (-2\sqrt{3} \cdot 0^2 \cdot (-4\sqrt{3})^3 + 2(-4\sqrt{3}))$$

$$= -8\sqrt{3} - 8\sqrt{3} = -16\sqrt{3}.$$

(8)



$$\vec{F} = 2x\hat{i}+2y\hat{j}+2z\hat{k} = 2(x\hat{i}+y\hat{j}+z\hat{k})$$

$$\vec{\nabla}\cdot\vec{F} = 2+2+2 = 6.$$

Gauss' divergence theorem gives

$$\iiint_{\Omega} \vec{\nabla}\cdot\vec{F}\, dV = \iint_{\partial\Omega} \vec{F}\cdot d\vec{S}$$

$$\iiint_{\Omega} \vec{\nabla}\cdot\vec{F}\, dV = \iiint_{\Omega} 6\, dV = 6\iiint_{\Omega} dV$$

$$= 6\,(\text{Volume of hemisphere})$$

$$= 6\,\frac{\frac{4}{3}\pi\cdot 1^3}{2} = \frac{8\pi}{2} = 4\pi$$

To verify Gauss' divergence theorem also compute $\iint_{\partial\Omega} \vec{F}\cdot d\vec{S}$.

Let $\partial\Omega_1$ be the hemisphere

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The hemisphere is parametrized by

$$\Phi_1(\varphi,\theta) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

and $\vec{T}_\varphi = (-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0)$

$\vec{T}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$

$$\vec{T}_\varphi \times \vec{T}_\theta = (-\sin^2\theta\cos\varphi, -\sin^2\theta\sin\varphi, -\sin\theta\cos\theta)$$

and the outward pointing normal is

$$\vec{n} = (\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \sin\theta\cos\theta)$$

$$\iint_{\partial\Omega_1} \vec{F}\cdot d\vec{S} = \iint_{\partial\Omega_1} 2(x\hat{i}+y\hat{j}+z\hat{k})\cdot(\sin^2\theta\cos\varphi, \sin^2\theta\sin\varphi, \sin\theta\cos\theta)\, d\theta d\varphi$$

$$= \iint_{\partial\Omega_1} 2(\sin^3\theta\cos^2\varphi + \sin^3\theta\sin^2\varphi + \sin\theta\cos^2\theta)\, d\theta d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\frac{\pi}{2}} 2(\sin^3\theta + \sin\theta\cos^2\theta)\, d\theta d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\frac{\pi}{2}} 2\sin\theta\, d\theta d\varphi = \int_{\varphi=0}^{\varphi=2\pi} -2\cos\theta\Big|_{\theta=0}^{\theta=\pi/2} d\varphi$$

$$= \int_{\varphi=0}^{\varphi=2\pi}(-2\cdot 0-(-2\cdot 1))\,d\varphi = 2\varphi\Big|_{\varphi=0}^{\varphi=2\pi} = 2\cdot 2\pi = 4\pi.$$

Q8

3

Let $\partial\Omega_2$ be the base of the hemisphere.

The base of the hemisphere is parametrized by

$$\Phi_2(\rho,\theta) = (\rho\cos\theta, \rho\sin\theta, 0)$$

and the outward pointing normal is $\hat{n} = -\hat{k}$.

Then $z=0$ on $\partial\Omega_2$ and

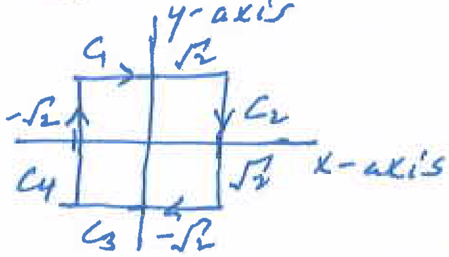
$$\iint_{\partial\Omega_2} \vec{F}\cdot d\vec{S} = \iint_{\partial\Omega_2} 2(x\hat{i}+y\hat{j}+0\hat{k})\cdot(-\hat{k})\,dS$$

$$= \iint_{\partial\Omega_2} 0\,dS = 0.$$

So $$\iint_{\partial\Omega} \vec{F}\cdot d\vec{S} = \iint_{\partial\Omega_1} \vec{F}\cdot dS + \iint_{\partial\Omega_2} \vec{F}\cdot d\vec{S} = 4\pi + 0 = 4\pi$$

(9) $\vec{F}(x,y,z) = \cos(xz)\,\hat{i} + x^3\,\hat{j} + y e^{-xz}\,\hat{k}$

(a)



$\vec{F}(x,\pm\sqrt{2},0) = \hat{i} + x^3\hat{j} \pm \sqrt{2}\,\hat{k}$

$\vec{F}(\pm\sqrt{2},y,0) = \hat{i} \pm 2\sqrt{2}\,\hat{j} + y\,\hat{k}$

$$d\vec{s} = \begin{cases} \hat{i}, & \text{if } y=\sqrt{2} \text{ on } C_1, \\ -\hat{j}, & \text{if } x=\sqrt{2} \text{ on } C_2, \\ -\hat{i}, & \text{if } y=-\sqrt{2} \text{ on } C_3, \\ \hat{j}, & \text{if } x=-\sqrt{2} \text{ on } C_4 \end{cases}$$

By Stokes theorem, $\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{s}$

$$= \int_{C_1} (\hat{i}+x^3\hat{j}+\sqrt{2}\hat{k})\cdot\hat{i}\,dx + \int_{C_2} (\hat{i}+2\sqrt{2}\hat{j}+y\hat{k})\cdot(-\hat{j})\,dy$$

$$+ \int_{C_3} (\hat{i}+x^3\hat{j}-\sqrt{2}\hat{k})\cdot(-\hat{i})\,dx + \int_{C_4} (\hat{i}-2\sqrt{2}\hat{j}+y\hat{k})\cdot\hat{j}\,dy$$

$$= \int_{x=-\sqrt{2}}^{x=\sqrt{2}} dx + \int_{y=-\sqrt{2}}^{y=\sqrt{2}} -2\sqrt{2}\,dy + \int_{x=-\sqrt{2}}^{x=\sqrt{2}} (-1)\,dx + \int_{y=-\sqrt{2}}^{y=\sqrt{2}} -2\sqrt{2}\,dy$$

$$= 2\int_{y=-\sqrt{2}}^{y=\sqrt{2}} -2\sqrt{2}\,dy = -4\sqrt{2}\,y\Big|_{y=-\sqrt{2}}^{y=\sqrt{2}} = -4\sqrt{2}\cdot 2\sqrt{2} = -16$$

(9b) Let S' be the cross-section of the cube at the $z=0$ plane (a square centred at $(0,0)$).

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos(xz) & x^3 & ye^{-xz} \end{vmatrix}$$

$$= \hat{i}(e^{-xz}-0) - \hat{j}(-zye^{-xz} + x\sin(xz)) + \hat{k}(3x^2-0)$$

$$= e^{-xz}\hat{i} + (zye^{-xz} - x\sin(xz))\hat{j} + 3x^2\hat{k}.$$

On the $z=0$ plane

$$\vec{\nabla}\times\vec{F} = e^0\hat{i} + (0 - x\sin 0)\hat{j} + 3x^2\hat{k} = \hat{i} + 3x^2\hat{k}$$

and $d\vec{S} = -\hat{k}$ (outward pointing normal).

So $(\vec{\nabla}\times\vec{F})\cdot d\vec{S} = (\hat{i}+3x^2\hat{k})\cdot(-\hat{k}) = -3x^2.$

By Stokes theorem,

$$\iint_S (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \iint_{S'} (\vec{\nabla}\times\vec{F})\cdot d\vec{S} = \int_{y=-\sqrt{2}}^{y=\sqrt{2}} \int_{x=-\sqrt{2}}^{x=\sqrt{2}} -3x^2\,dx$$

$$= \int_{y=-\sqrt{2}}^{y=\sqrt{2}} -x^3 \Big|_{x=-\sqrt{2}}^{x=\sqrt{2}} dy = \int_{y=-\sqrt{2}}^{y=\sqrt{2}} (-2\sqrt{2}-2\sqrt{2})\,dy$$

$$= -4\sqrt{2}\,y\Big|_{y=-\sqrt{2}}^{y=\sqrt{2}} = -4\sqrt{2}(\sqrt{2}-(-\sqrt{2})) = -4\sqrt{2}\cdot 2\sqrt{2}$$

$$= -16.$$

(10) (a) Green's Theorem states

$$\int_{C=\partial D} P\,dx + Q\,dy = \int_{\partial D} \vec{F}\cdot d\vec{s} = \iint_D (\vec{\nabla}\times\vec{F})\,\hat{k}\,dA = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

where $\vec{F} = P\hat{i} + Q\hat{j}$ is a C^1-vector field on D

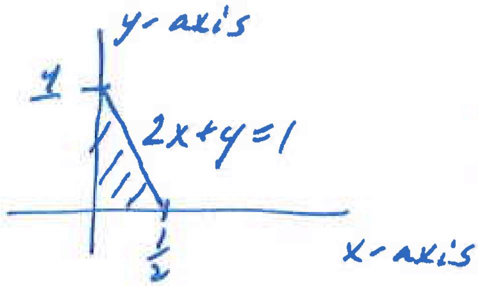
D is a region in the xy plane

bounded by a simple closed curve $C = \partial D$

which is oriented in the positive direction

(counterclockwise).

(b)



$\int_C y^2\,dx + x^2\,dy$

has $\vec{F} = y^2\hat{i} + x^2\hat{j}$.

Green's theorem gives

$$\int_C y^2\,dx + x^2\,dy = \iint_D \left(\frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\right) dx\,dy$$

$$= \int_{x=0}^{x=\frac{1}{2}} \int_{y=0}^{y=1-2x} (2x-2y)\,dy\,dx = \int_{x=0}^{x=\frac{1}{2}} \left((2xy - y^2)\Big|_{y=0}^{y=1-2x} \right) dx$$

$$= \int_{x=0}^{x=\frac{1}{2}} \left(2x(1-2x) - (1-2x)^2\right) dx$$

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(2)

$$= \int_{x=0}^{x=\frac{1}{2}} (2x - (1-2x))(1-2x)\, dx$$

$$= \int_{x=0}^{x=\frac{1}{2}} (4x-1)(1-2x)\, dx = \int_{x=0}^{x=\frac{1}{2}} (-8x^2+6x-1)\, dx$$

$$= -\frac{8}{3}x^3 + 3x^2 - x \Big|_{x=0}^{x=\frac{1}{2}} = -\frac{8}{3}\frac{1}{8} + 3\cdot\frac{1}{4} - \frac{1}{2} - (0+0-0)$$

$$= -\frac{1}{3} + \frac{3}{4} - \frac{1}{2} = -\frac{1}{3} + \frac{1}{4} = \frac{-1}{12}$$

(11) $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ with $x = \frac{1}{2}(u^2 - v^2)$

$y = uv$

$z = z$

(a)

$\frac{\partial \vec{r}}{\partial u} = (u, v, 0)$ $\quad h_u = \left|\frac{\partial \vec{r}}{\partial u}\right| = \sqrt{u^2+v^2+0^2} = \sqrt{u^2+v^2}$

$\frac{\partial \vec{r}}{\partial v} = (-v, u, 0)$ $\quad h_v = \left|\frac{\partial \vec{r}}{\partial v}\right| = \sqrt{(-v)^2+u^2+0^2} = \sqrt{u^2+v^2}$

$\frac{\partial \vec{r}}{\partial z} = (0, 0, 1)$ $\quad h_z = \left|\frac{\partial \vec{r}}{\partial z}\right| = \sqrt{0^2+0^2+1^2} = 1.$

(b) Since

$(u, v, 0) \cdot (-v, u, 0) = -uv + vu + 0 = 0,$

$(u, v, 0) \cdot (0, 0, 1) = 0 + 0 + 0 = 0,$

$(-v, u, 0) \cdot (0, 0, 1) = 0 + 0 + 0 = 0,$

the coordinate system is orthogonal.

(c) Jacobian $= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} u & -v & 0 \\ v & u & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$= u^2 + v^2$

Note that $u^2 + v^2 = h_u \cdot h_v \cdot h_z$.

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(d) $f(u,v,z) = u^3v^5 + 5678z^2 + 3456$

$\vec{F}(u,v,z) = u^2 e_u$

Then

$$\vec{\nabla} f = \frac{1}{h_u}\frac{\partial f}{\partial u}\hat{e}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\hat{e}_v + \frac{1}{h_z}\frac{\partial f}{\partial z}\hat{e}_z$$

$$= \frac{1}{\sqrt{u^2+v^2}} 3u^2v^5\,\hat{e}_u + \frac{1}{\sqrt{u^2+v^2}} 5u^3v^4\,\hat{e}_v + \frac{1}{1} 5678\cdot 2z\,\hat{e}_z$$

$$= \frac{1}{u^2+v^2} 3u^2v^5(u,v,0) + \frac{1}{u^2+v^2} 5u^3v^4(-v,u,0) + 5678\cdot 2z(0,0,1)$$

$$= \frac{3u^3v^5 - 5u^3v^5}{u^2+v^2}\hat{\imath} + \frac{3u^2v^6 + 5u^4v^4}{u^2+v^2}\hat{\jmath} + 11356z\hat{k}$$

$$= \frac{-2u^3v^5}{u^2+v^2}\hat{\imath} + \frac{u^2v^4(3v^2+5u^2)}{u^2+v^2}\hat{\jmath} + 11356z\hat{k}.$$

$$\vec{\nabla} \times \vec{F} = \frac{1}{h_u h_v h_z} \begin{vmatrix} h_u \hat{e}_u & h_v \hat{e}_v & h_z \hat{e}_z \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ h_u F_1 & h_v F_2 & h_z F_3 \end{vmatrix}$$

where $\vec{F} = u^2 \hat{e}_u$. So

$$\vec{\nabla} \times \vec{F} = \frac{1}{u^2+v^2} \begin{vmatrix} \hat{e}_u \sqrt{u^2+v^2} & \hat{e}_v \sqrt{u^2+v^2} & \hat{e}_z \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial z} \\ \sqrt{u^2+v^2}\, u^2 & 0 & 0 \end{vmatrix}$$

$$= \frac{1}{u^2+v^2} \Big(\hat{e}_u \sqrt{u^2+v^2} (0-0) - \hat{e}_v \sqrt{u^2+v^2} (0-0)$$
$$+ \hat{e}_z \big(0 - u^2 \tfrac{1}{2} (u^2+v^2)^{-\frac{1}{2}} 2v\big) \Big)$$

$$= \frac{1}{u^2+v^2} \frac{-u^2 v}{\sqrt{u^2+v^2}} \hat{e}_z = \frac{-u^2 v}{(u^2+v^2)^{3/2}} \hat{e}_z$$