Vector Calculus Lecture 4

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UniMelb

(1)

The derivative matrix of f is the $m \times n$ matrix

$$D(f) = \left(\frac{\partial f_i}{\partial x_j}\right) \quad \text{for} \quad \begin{aligned} f: \mathbb{R}^m &\to \mathbb{R}^n \\ (x_1, \dots, x_m) &\mapsto (f_1, \dots, f_n) \end{aligned}$$

The Jacobian of f is $\det(D(f))$.

Theorem (Chain rule) Let $a = (a_1, \dots, a_m) \in \mathbb{R}^m$.
If $f: \mathbb{R}^m \to \mathbb{R}^n$ is C^1 at a and
$g: \mathbb{R}^n \to \mathbb{R}^p$ is C^1 at $f(a)$ then

$$D(g \circ f)\big|_a = \left(D(g)\big|_{f(a)}\right)\left(D(f)\big|_a\right).$$

§1.4 Example 1: Let $h = f(u,v,w)$ where

$$u = u(x,y,z),$$
$$v = v(x,y,z),$$
$$w = w(x,y,z).$$

Find $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial y}$, $\frac{\partial h}{\partial z}$.

Solution: By the chain rule for partial derivatives:

$$\frac{\partial h}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x},$$

$$\frac{\partial h}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y},$$

$$\frac{\partial h}{\partial z} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}.$$

In matrix form this is

$$\begin{pmatrix} \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial v} & \frac{\partial f}{\partial w} \end{pmatrix} \frac{\partial(u,v,w)}{\partial(x,y,z)}.$$

§1.4 Example 2 Find the derivative matrix for the function f defined by

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \quad \text{with} \quad f(u,v) = (u+v,\ u,\ v^2).$$

Solution:

$$\underline{\underline{D}}f = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}$$

§1.4 Example 3 Find the derivative matrix for the function $g : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$g(x,y) = (x^2+1,\ y^2).$$

Solution:

$$\underline{\underline{D}}(g) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

§1.4 Example 4 Let $f: \mathbb{R}^2 \to \mathbb{R}^3$ and $g: \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(u,v) = (u+v, u, v^2) \text{ and } g(x,y) = (x^2+1, y^2)$$

Compute $\underline{D}(f \circ g)$ at $(x,y) = (1,1)$.

Solution:

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$$

$$(x,y) \longmapsto (x^2+1, y^2)$$

$$(u,v) \longmapsto (u+v, u, v^2)$$

$$(1,1) \longmapsto (2,1) \longmapsto (3,2,1)$$

$$\underline{D}(f) = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix}$$

$$\underline{D}(g) = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix}$$

By the chain rule

$$\underline{D}(f \circ g)\Big|_{(x,y)=(1,1)} = \left(\underline{D}(f)\Big|_{(u,v)=(2,1)}\right)\left(\underline{D}(g)\Big|_{(x,y)=(1,1)}\right)$$

$$= \left(\left. \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2v \end{pmatrix} \right|_{(u,v)=(2,1)} \right) \left(\left. \begin{pmatrix} 2x & 0 \\ 0 & 2y \end{pmatrix} \right|_{(x,y)=(1,1)} \right)$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Alternatively,

$$(f \circ g)(x,y) = f(x^2+1, y^2) = (x^2+1+y^2, x^2+1, y^4)$$

so

$$\underset{\sim}{D}(f \circ g) = \begin{pmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} \\ \frac{\partial h_3}{\partial x} & \frac{\partial h_3}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ 2x & 0 \\ 0 & 4y^3 \end{pmatrix}$$

and

$$\underset{\sim}{D}(f \circ g) \Big|_{(x,y)=(1,1)} = \left. \begin{pmatrix} 2x & 2y \\ 2x & 0 \\ 0 & 4y^3 \end{pmatrix} \right|_{(x,y)=(1,1)} = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 0 & 4 \end{pmatrix}$$

§1.4 Example (Polar coordinates)

$x = r\cos\theta$,
$y = r\sin\theta$. Find the Jacobian.

Solution:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

So

$$\text{Jacobian} = \det\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right) = r\cos^2\theta - (-r\sin^2\theta)$$

$$= r(\cos^2\theta + \sin^2\theta) = r.$$

31.4 Example (Cylindrical coordinates)

$x = \rho \cos\varphi,$

$y = \rho \sin\varphi,$ Find the Jacobian.

$z = z.$

Solution:

$$\text{Jacobian} = \det\left(\frac{\partial(x,y,z)}{\partial(\rho,\varphi,z)}\right)$$

$$= \det\begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{pmatrix} = \det\begin{pmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= 0 - 0 + 1\cdot(\cos^2\varphi\cdot\rho - \sin\varphi(-\rho\sin\varphi))$$

$$= \rho\cos^2\varphi + \rho\sin^2\varphi = \rho.$$

51.4 Example (Spherical coordinates) UniMelb

$x = r\sin\theta\cos\varphi,$

$y = r\sin\theta\sin\varphi,$ Find the Jacobian.

$z = r\cos\theta.$

Solution:

$$\text{Jacobian} = \det\left(\frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}\right) = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{pmatrix}$$

$$= \det\begin{pmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{pmatrix}$$

$$= \cos\theta\left(r^2\sin\theta\cos\theta\cos^2\varphi - (-r^2\sin\theta\cos\theta\sin^2\varphi)\right)$$
$$-(-r\sin\theta)\left(r\sin^2\theta\cos^2\varphi - (-r\sin^2\theta\sin^2\varphi)\right)$$
$$+0$$

$$= \cos\theta\left(r^2\sin\theta\cos\theta(\cos^2\varphi + \sin^2\varphi)\right)$$
$$+ r\sin\theta\left(r\sin^2\theta(\cos^2\varphi + \sin^2\varphi)\right)$$

$$= r^2\sin\theta\cos^2\theta + r^2\sin\theta\sin^2\theta$$

$$= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta$$