Vector calculus Lecture 22

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Curves $\vec{c}(t) = (x(t), y(t), z(t))$

$$\frac{d\vec{c}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$d\vec{s} = \frac{d\vec{c}}{dt}\,dt \quad \text{and} \quad ds = \left|\frac{d\vec{c}}{dt}\right| dt$$

Integrals:

$$\int_{\vec{c}} f\,ds \quad \text{and} \quad \int_{\vec{c}} \vec{F}\cdot d\vec{s}$$

total of f along $\vec{c}$ — Work done by $\vec{F}$ along $\vec{c}$

Surfaces $\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$

$$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) \quad \text{and} \quad \vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$$

$$d\vec{S} = \vec{T}_u \times \vec{T}_v\,du\,dv \quad \text{and} \quad dS = |\vec{T}_u \times \vec{T}_v|\,du\,dv$$

Integrals:

$$\iint_S f\,dS \quad \text{and} \quad \iint_S \vec{F}\cdot d\vec{S}$$

total of f across S — Flux of $\vec{F}$ across S

§4.3 Example 1 Identify and graph the surface given by

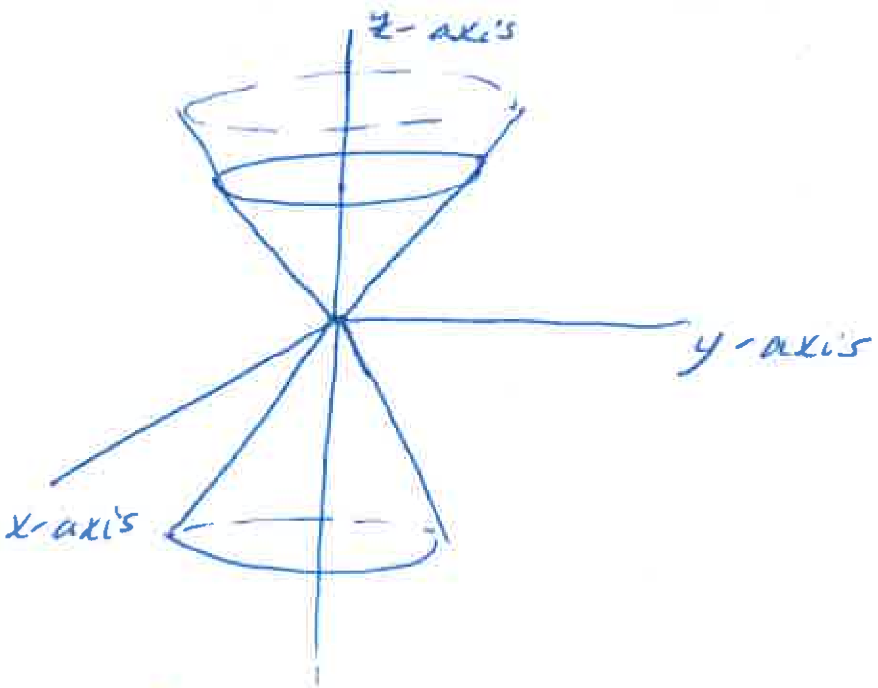
$\Phi(u,v) = (u\cos v, u\sin v, u)$

for $0 < v \leq 2\pi$, $u \geq 0$.

Solution Since $x = u\cos v$ and $y = u\sin v$ and $z = u$ then

$$x^2 + y^2 = u^2\cos^2 v + u^2 \sin^2 v = u^2 = z^2.$$

For a fixed value of z (height) the graph is a circle. As z increases the radius of the circle increases.



This surface is a cone.

§4.3 Example 2 Identify and graph the surface given by

$$\Phi(u,v) = ((R+\cos v)\cos u, (R+\cos v)\sin u, \sin v)$$

where R is fixed, $R>0$ and

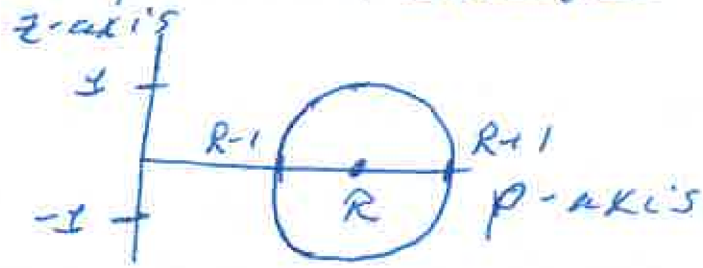
$$0 \le u \le 2\pi \quad \text{and} \quad 0 \le v \le 2\pi.$$

Solution Write

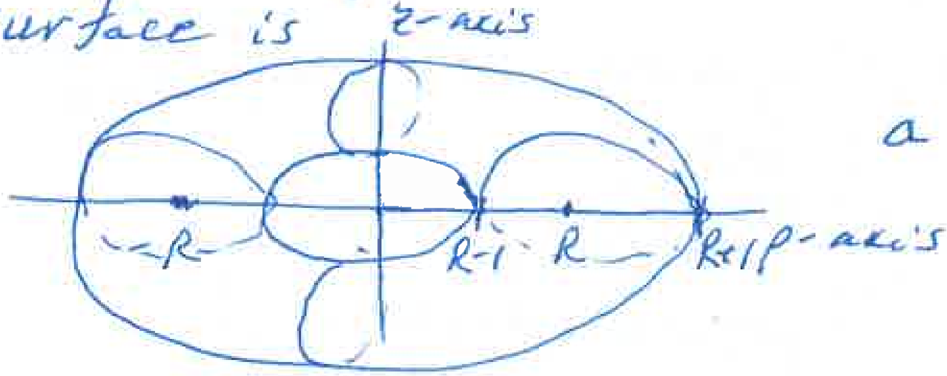
$x = (R+\cos v)\cos u$
$y = (R+\cos v)\sin u$ in cylindrical coordinates
$z = \sin v$

to get $\rho = R + \cos v$ and $\varphi = u$.
$z = \sin v$.

This in the ρz-coordinates



(the circle $\rho = \cos v$, $z = \sin v$, but with ρ shifted by R). As $\varphi = u$ runs from 0 to 2π the surface is

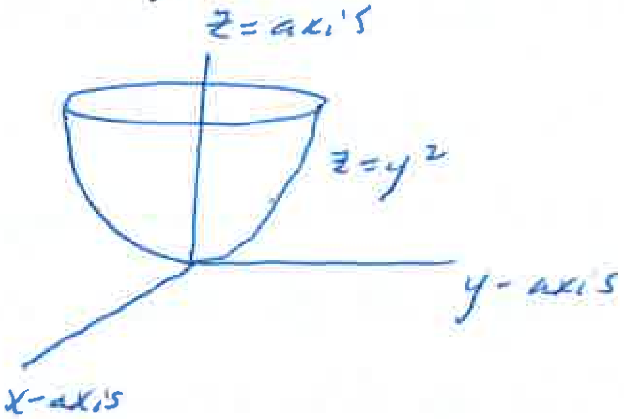


a torus (doughnut)

§ 4.3 Example 3 Parametrise the surface of the parabolic bowl.

$$z = x^2 + y^2.$$

Solution



One parametrisation is in u, v variables.

$$\Phi(u,v) = (u, v, u^2+v^2) \qquad \begin{array}{l} -\infty < u < \infty \\ -\infty < v < \infty \end{array}$$

so that $x=u$, $y=v$ and $z=x^2+y^2$.

With this parametrization, then cylindrical coordinates

$$\rho = \sqrt{u^2+v^2} = \sqrt{z}$$
$$\varphi = \varphi \qquad \text{with} \qquad 0 \le \varphi \le 2\pi$$
$$z = \rho^2 \qquad\qquad\qquad 0 \le z < \infty$$

gives a parametrization in φ and z variables.

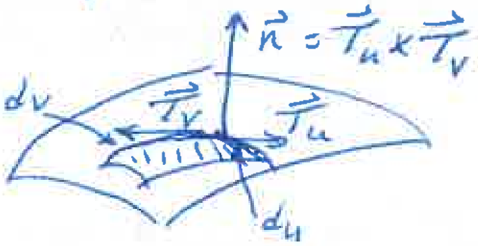
If

$$\Phi(u,v) = (x(u,v), y(u,v), z(u,v))$$

describes a surface then the tangent vectors along u and along v are

$$\vec{T}_u = \left(\frac{\partial x}{\partial u} \quad \frac{\partial y}{\partial u} \quad \frac{\partial z}{\partial u}\right)$$

$$\vec{T}_v = \left(\frac{\partial x}{\partial v} \quad \frac{\partial y}{\partial v} \quad \frac{\partial z}{\partial v}\right)$$



and ~~their~~ cross product

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}$$ is a normal vector to the surface

and $\hat{n} = \dfrac{\vec{T}_u \times \vec{T}_v}{|\vec{T}_u \times \vec{T}_v|}$ is a unit normal vector to the surface.

If $\vec{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ is a vector in $\mathbb{R}^3$ and $(a,b,c) \in \mathbb{R}^3$ then

$$(x-a, y-b, z-c) \cdot (n_1, n_2, n_3) = 0$$

gives

$$n_1 x + n_2 y + n_3 z = n_1 a + n_2 b + n_3 c$$

as the equation of a plane in $\mathbb{R}^3$ through the point (a,b,c) and perpendicular to the vector $\vec{n}$.

§4.3 Example 4 Find a normal vector to the cone parametrized by

$$\Phi(u,v) = (u\cos v, u\sin v, u)$$

for $0 \le v \le 2\pi$ and $0 \le u < \infty$.

Solution The tangent vectors are

$$\vec{T}_u = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right) = (\cos v, \sin v, 1)$$

$$\vec{T}_v = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right) = (-u\sin v, u\cos v, 0)$$

and

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \begin{matrix} \hat{\imath}(0 - u\cos v) \\ -\hat{\jmath}(0 + u\sin v) \\ +\hat{k}(u\cos^2 v + u\sin^2 v) \end{matrix}$$

$$= (-u\cos v, -u\sin v, u).$$

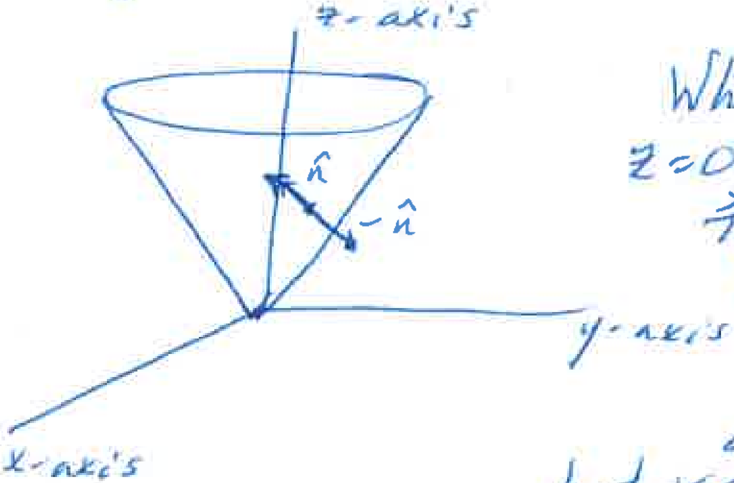
The unit normal vector to the surface is

$$\hat{n} = \frac{\vec{T}_u \times \vec{T}_v}{|\vec{T}_u \times \vec{T}_v|} = \frac{(-u\cos v, -u\sin v, u)}{\sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2}}$$

$$= \frac{1}{\sqrt{2u^2}}(-u\cos v, -u\sin v, u) \qquad (\text{for } u \neq 0)$$

$$= \frac{1}{\sqrt{2}}(-\cos v, -\sin v, 1)$$

This vector is "inward pointing" and $-\hat{n} = \frac{1}{\sqrt{2}}(\cos v, \sin v, -1)$ is "outward pointing".



When $u=0$ then $z=0$ and $\vec{T}_u \times \vec{T}_v$ has length 0. The cone is <u>not</u> <u>smooth</u> at this point but is smooth at all other points.

<u>§4.3 Example 5</u> For the cone of Example 4 find the tangent plane to the cone at $(1,1,\sqrt{2})$.

<u>Solution</u> Since

$$x = u\cos v$$
$$y = u \sin v$$
$$z = u$$

then, when $(x,y,z) = (1,1,\sqrt{2})$,

$$u = \sqrt{2} \text{ and } v = \arctan\left(\frac{y}{x}\right) = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}.$$

So the normal vector to the cone at $(1,1,\sqrt{2})$ is

$$\vec{T}_u \times \vec{T}_v \Big|_{\substack{(x,y,z)\\=(1,1,\sqrt{2})}} = \vec{T}_u \times \vec{T}_v \Big|_{\substack{(u,v)\\=(\sqrt{2},\frac{\pi}{4})}} =$$

$$= \left(-\sqrt{2}\cos\tfrac{\pi}{4}, -\sqrt{2}\sin\tfrac{\pi}{4}, \sqrt{2}\right)$$
$$= (-1, -1, \sqrt{2})$$

The equation of the plane tangent to the cone at $(x,y,z)=(1,1,\sqrt{2})$ is

$$(x-1,\ y-1,\ z-\sqrt{2})\cdot(-1,-1,\sqrt{2})=0$$

which is

$$(-x+1)+(-y+1)+(\sqrt{2}\,z-2)=0$$

which is

$$x+y-\sqrt{2}\,z=0.$$