Algebraic Geometry Week 7 05.09.2018 ①
Tangent space and differential forms UniMelb
A. Ram

The Koszul complex or de Rham complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{top} \to 0$$

is a complex of $\mathcal{O}_X$-modules.

$\mathcal{O}_X$ = ring of functions on X

T_X = tangent bundle

$\Omega_X^1 = T_X^*$ = cotangent bundle
= module of differentials

$\Omega_X^p = \Lambda^p \Omega_X^1$ = module of p-forms

Ω_X^{top} = canonical bundle.

If $X = \mathrm{Spec}(\mathbb{C}[x_1, \ldots, x_n])$, the sections of

T_X have $\mathcal{O}_X$-basis $\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\}$

Ω_X^1 have $\mathcal{O}_X$-basis $\{dx_1, \ldots, dx_n\}$

Ω_X^p have $\mathcal{O}_X$-basis $\left\{dx_{i_1} \wedge \cdots \wedge dx_{i_p} \;\middle|\; \begin{matrix} i_1, \ldots, i_p \in \{1, \ldots, n\} \\ i_1 < \cdots < i_p \end{matrix}\right\}$

$\mathcal{O}_X$ has $\mathcal{O}_X$-basis $\{1\}$

(see Harder §4.3.3 p.63 and §4.10.2 p.156)

(see also §7.5.3 Examples 15, 16)

Alg. Geom. Week 7 05.09.2018

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7

The Koszul complex

The exterior algebra

$\Omega_X = \Lambda(\Omega^1_X)$ is an $\mathcal{O}_X$-algebra with $\wedge$ as a product. Then Ω_X is a graded algebra $\Omega_X = \bigoplus_{p\in\mathbb{Z}_{\geq 0}} \Omega^p_X$ with

$\Lambda^0(\Omega^1_X) = \mathcal{O}_X$, $\Lambda^1(\Omega^1_X) = \Omega^1_X$ and $\Omega^p_X = \Lambda^p(\Omega^1_X)$.

The Koszul complex

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega^1_X \xrightarrow{d} \Omega^2_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{top}_X \to 0$$

is obtained by extending $\mathcal{O}_X \xrightarrow{d} \Omega^1_X$ by

$d^2 = 0$ and

$$d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^{\deg(\omega_1)\deg(d)} \omega_1 \wedge d\omega_2$$

where $\deg(d) = 1$. (Harder §4.10.1 p 151 and Bourbaki Alg. Ch. X, Algèbre Homologique Prop 13).

HW Look up the definition of differential graded algebra (dg algebra) and show that (Ω_X, d) is a dg algebra.

The cotangent bundle Ω^1_X is (Harder §7.5.2)

$$\Omega^1_X = \frac{\mathcal{I}}{\mathcal{I}^2} \quad \text{where } \mathcal{I} = \ker\begin{pmatrix} \mathcal{O}_X \otimes_k \mathcal{O}_X \xrightarrow{\mathrm{Spec}(\Delta)} \mathcal{O}_X \\ f_1 \otimes f_2 \longmapsto f_1 f_2 \end{pmatrix}$$

with the k-module map

$$d: \mathcal{O}_X \to \Omega^1_X \text{ given by } df = f \otimes 1 - 1 \otimes f.$$

Since $\mathcal{I}$ is an $\mathcal{O}_X$-module then Ω^1_X is an $\mathcal{O}_X$-module. The map $d: \mathcal{O}_X \to \Omega^1_X$ is surjective (see Harder, Lemma 7.5.7).

HW: Show that the map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X) & \longrightarrow & \left\{\begin{matrix}\text{derivations} \\ \text{of } \mathcal{O}_X\end{matrix}\right\} \\ \omega & \longmapsto & \omega \circ d \end{array}$$

is an isomorphism of $\mathcal{O}_X$-modules, so that the tangent bundle T_X is the dual of Ω^1_X. (see Harder §7.5.5).

HW: (The chain rule).

A morphism $\varphi: X \to Y$ corresponds to a morphism $\varphi^*: \mathcal{O}_Y \to \mathcal{O}_X$ given by $\varphi^*(f) = f \circ \varphi$. Then there is $T_\varphi: T_X \to T_Y$ and $T_{\psi\circ\varphi} = T_\psi \circ T_\varphi$, so that T is a "functor".

Constructing the cotangent bundle Ω_X

The module of differentials, is the pair (Ω_X^1, d) where Ω_X^1 is an $\mathcal{O}_X$-module and $d: \mathcal{O}_X' \to \Omega_X^1$ is a $\mathbb{C}$-linear map such that

$$d(f_1 f_2) = f_1(df_2) + f_2(df_1)$$

which satisfies the following universal property:

If M is a $\mathcal{O}_X$-module and $d_1: \mathcal{O}_X \to M$ is a $\mathbb{C}$-linear map such that

$$d_1(f_1 f_2) = f_1(d_1 f_2) + \cancel{f_2}(d_1 f_1)$$

then there exists a unique $\mathcal{O}_X$-module morphism $\varphi: \Omega_X^1 \to M$ with $\varphi \circ d = d_1$

$$\begin{tikzcd} \mathcal{O}_X \arrow[rr, "d_1"] \arrow[dr, "d"'] & & M \\ & \Omega_X^1 \arrow[ur, "\varphi"'] & \end{tikzcd}$$

(see Harder Prop. 7.5.10(a))

Existence of (Ω_X^1, d): The diagonal map is

$$\begin{aligned} X &\xrightarrow{\Delta} X \times X \\ p &\longmapsto (p,p) \end{aligned} \quad \text{and} \quad \begin{aligned} \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X &\xrightarrow{\mathrm{Spec}(\Delta)} \mathcal{O}_X \\ f_1 \otimes f_2 &\longmapsto f_1 f_2 \end{aligned}$$

Then $\mathrm{Spec}(\Delta)$ is an $\mathcal{O}_X$-module map where $f(g_1 \otimes g_2) = fg_1 \otimes g_2$ specifies the $\mathcal{O}_X$-action on $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_X$.

<u>The Weyl algebra</u> Let $[a,b]=ab-ba$.

Let $X=\mathrm{Spec}(\mathbb{C}[x_1,\ldots,x_n])$.

The <u>symmetric algebra</u> is

$$\mathcal{O}_X=\mathbb{C}[x_1,\ldots,x_n],$$ the $\mathbb{C}$-algebra generated by $x_1,\ldots,x_n$ with relations

$$[x_i,x_j]=0 \quad \text{for } i,j\in\{1,\ldots,n\}$$

The <u>Weyl algebra</u>, or <u>Heisenberg algebra</u>, is

$$\mathcal{D}_X=\mathbb{C}[x_1,\ldots,x_n]\otimes\mathbb{C}\left[\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right],$$

the $\mathbb{C}$-algebra generated by $x_1,\ldots,x_n,\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}$ with relations

$$[x_i,x_j]=0,\quad \left[\frac{\partial}{\partial x_i},\frac{\partial}{\partial x_j}\right]=0,\quad \left[\frac{\partial}{\partial x_j},x_i\right]=\delta_{ij}.$$

The <u>exterior algebra</u> is

$$\Omega_X=\Lambda(\Omega^1_X),$$

the $\mathbb{C}$-algebra with product $\wedge$, generated by $dx_1,\ldots,dx_n$ with relations

$$dx_i\wedge dx_j=-dx_j\wedge dx_i,$$

for $i,j\in\{1,\ldots,n\}$.

Normally one identifies

$$\{\text{points of } X\} \longleftrightarrow \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathcal{O}_X, \mathbb{C})$$
$$p \longmapsto ev_p$$

where $ev_p(f) = f(p)$. With this notation

$$\gamma = ev_p + t\eta \qquad (\text{and } \xi = ev_p)$$

and
$$\eta(f_1 f_2) = f_1(p)\eta(f_2) + \eta(f_1)f_2(p).$$

A <u>vector field</u>? is a section ∂ of T_X, i.e. a choice of tangent vector at each point $p \in X$. So

$$\partial : \mathcal{O}_X \to \mathcal{O}_X \quad \text{is a } \mathbb{C}\text{-linear map}$$

such that
$$\partial(f_1 f_2) = f_1(\partial f_2) + (\partial f_1) f_2.$$

Hence
$$\left\{\begin{matrix}\text{derivations}\\ \text{of } \mathcal{O}_X\end{matrix}\right\} = \left\{\begin{matrix}\text{vector fields}\\ \text{on } X\end{matrix}\right\} = \left\{\begin{matrix}\text{sections}\\ \text{of } T_X\end{matrix}\right\}$$

Constructing the tangent bundle T_X

Let $\mathcal{O}_X$ be a ~~ring~~. $\mathbb{C}$-algebra

A *derivation* is a $\mathbb{C}$-linear map $\partial : \mathcal{O}_X \to \mathcal{O}_X$ such that

$$\partial(f_1 f_2) = f_1(\partial f_2) + (\partial f_1) f_2 .$$

Let $(X, \mathcal{T}_X, \mathcal{O}_X)$ be a ringed space.

Let $p \in X$. A *tangent vector to X at p* is a $\mathbb{C}$-linear map $\eta : \mathcal{O}_X \to \mathbb{C}$ such that

$$\eta(f_1 f_2) = f_1(p)\eta(f_2) + \eta(f_1) f_2(p)$$

for $f_1, f_2 \in \mathcal{O}_X$. The *tangent bundle to X* is

$$T_X = \mathrm{Hom}_{\mathbb{C}\text{-alg}}\left(\mathcal{O}_X, \frac{\mathbb{C}[t]}{t^2\mathbb{C}[t]}\right) \quad \text{with} \quad \begin{array}{c} T_X \\ \downarrow{\scriptstyle t=0} \\ X \end{array}$$

If $\gamma \in T_X$ and $\xi : \mathcal{O}_X \to \mathbb{C}$ and $\eta : \mathcal{O}_X \to \mathbb{C}$ are such that

$$\gamma = \xi + t\eta$$

(see Harder equation (7.19))

then
$$\xi(f_1 f_2) = \xi(f_1)\xi(f_2)$$

and
$$\eta(f_1 f_2) = \xi(f_1)\eta(f_2) + \eta(f_1)\xi(f_2)$$

Alg. Geom Week 7

05.09.2018 UniHelb A. Kaw ⑧

Connections

Let $\mathcal{M}$ be a vector bundle on X (i.e. a locally free $\mathcal{O}_X$-module).

A connection on $\mathcal{M}$ is a $\mathbb{C}$-module morphism

$$\nabla : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X^1$$

such that

$$\nabla(mf) = \nabla(m) f + m\, df$$

for $f \in \mathcal{O}_X$ and $m \in \mathcal{M}$. Define

$$0 \to \mathcal{M} \xrightarrow{\nabla} \mathcal{M} \otimes \Omega_X^1 \xrightarrow{\nabla} \mathcal{M} \otimes \Omega_X^2 \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{M} \otimes \Omega_X^{top} \to 0$$

by

$$\nabla(m \otimes \omega) = m \otimes d\omega + \nabla(m) \wedge d\omega$$

for $m \in \mathcal{M}$ and $\omega \in \Omega_X^p$.

The connection ∇ is flat if

$$\nabla^2 = 0.$$

Coordinates for $M\otimes\Omega_X^1$

Let M be a vector bundle on X (i.e. a locally free $\mathcal{O}_X$-module).
Choose a (local) basis of M

$$e_1, e_2, \ldots, e_n \quad \text{so that} \quad M = \mathcal{O}_X^{\oplus n}$$

is (locally) column vectors of length n with entries in $\mathcal{O}_X$; if $m \in M$ then

$$m = e_1 f_1 + \cdots + e_n f_n = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$ with respect to the basis $e_1, \ldots, e_n$.

Let $\nabla : M \to M \otimes_{\mathcal{O}_X} \Omega_X^1$ be a connection and define $A_{ij} \in \Omega_X^1$ by

$$\nabla(e_i) = e_1 \otimes A_{1i} + \cdots + e_n \otimes A_{ni}.$$

Then

$$\nabla(m) = \nabla(f_1 e_1) + \cdots + \nabla(f_n e_n)$$

$$= \nabla(e_1) f_1 + e_1 df_1 + \cdots + \nabla(e_n) f_n + e_n df_n$$

$$= e_1 \otimes A_{11} f_1 + \cdots + e_n A_{n1} f_1 + e_1 df_1$$
$$+ e_2 \otimes A_{12} f_2 + \cdots + e_n A_{n2} f_2 + e_2 df_2$$
$$+ \cdots +$$
$$+ e_n \otimes A_{1n} f_n + \cdots + e_n \otimes A_{nn} f_n + e_n df_n$$

$$= e_1(df_1 + A_{11}f_1 + A_{12}f_2 + \cdots + A_{1n}f_n)$$
$$+ e_2(df_2 + A_{21}f_1 + A_{22}f_2 + \cdots + A_{2n}f_n)$$
$$+ \cdots + e_n(df_n + A_{n1}f_1 + A_{n2}f_2 + \cdots + A_{nn}f_n)$$

$$= \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix} + \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

$$= (d + A)\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \quad \text{where } A = \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix}$$

is an $n\times n$ matrix with entries in Ω_X^1

and $d\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = \begin{pmatrix} df_1 \\ \vdots \\ df_n \end{pmatrix}$

Let $\omega_i = df_i + f_1 A_{i1} + f_2 A_{i2} + \cdots + f_n A_{in}$

so that

$$\nabla(m) = e_1 \otimes \omega_1 + \cdots + e_n \otimes \omega_n.$$

Then

$$\nabla^2(m) = \nabla(\nabla(m)) = \nabla(e_1 \otimes \omega_1 + \cdots + e_n \otimes \omega_n)$$

$$= \nabla(e_1)\wedge\omega_1 + e_1 \otimes d\omega_1 + \cdots + \nabla(e_n)\wedge\omega_n + e_n \otimes d\omega_n$$

Alg. Geom. Week 7 05.09.2018
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A. Ram

$$= e_1 \otimes A_{11} \wedge \omega_1 + \cdots + e_n \otimes A_{n1} \wedge \omega_1 + e_1 \otimes d\omega_1$$

$$+ e_1 \otimes A_{12} \wedge \omega_2 + \cdots + e_n \otimes A_{n2} \wedge \omega_2 + e_2 \otimes d\omega_2$$

$$+ \cdots$$

$$+ e_1 \otimes A_{1n} \wedge \omega_n + \cdots + e_n \otimes A_{nn} \wedge \omega_n + e_n \otimes d\omega_n$$

$$= e_1 \otimes (A_{11} \wedge \omega_1 + A_{12} \wedge \omega_2 + \cdots + A_{1n} \wedge \omega_n + d\omega_1)$$

$$+ e_2 \otimes (A_{21} \wedge \omega_1 + A_{22} \wedge \omega_2 + \cdots + A_{2n} \wedge \omega_n + d\omega_2)$$

$$+ \cdots$$

$$+ e_n \otimes (A_{n1} \wedge \omega_1 + A_{n2} \wedge \omega_2 + \cdots + A_{nn} \wedge \omega_n + d\omega_n).$$

Since $\omega_i = df_i + f_1 A_{i1} + \cdots + f_n A_{in}$ then

$$d\omega_i = d^2 f_i + df_1 \wedge A_{i1} + f_1 \, dA_{i1} + \cdots + df_n \wedge A_{in} + f_n \, dA_{in}$$

$$= 0 + \sum_{j=1}^{n} df_j \wedge A_{ij} + \sum_{k=1}^{n} dA_{ik} f_k$$

and

$$\sum_{j=1}^{n} A_{ij} \wedge \omega_j = \sum_{j=1}^{n} A_{ij} \wedge (df_j + f_1 A_{j1} + \cdots + f_n A_{jn})$$

$$= \left(-\sum_{j=1}^{n} df_j \wedge A_{ij}\right) + \sum_{j,k=1}^{n} A_{ij} \wedge A_{jk} f_k$$

So

$$\nabla^2(m) = \sum_{i=1}^{n} e_i \otimes (A_{i1} \wedge \omega_1 + \cdots + A_{in} \wedge \omega_n + d\omega_i)$$

$$= \sum_{i=1}^{n} e_i \otimes \left(-\sum_{j=1}^{n} df_j \wedge A_{ij} + \sum_{j,k=1}^{n} A_{ij} \wedge A_{jk} f_k + \sum_{j=1}^{n} df_j \wedge A_{ij} + \sum_{k=1}^{n} dA_{ik} f_k \right)$$

$$= \sum_{i=1}^{n} e_i \otimes \left(\sum_{k=1}^{n} \left(dA_{ik} + \sum_{j=1}^{n} A_{ij} \wedge A_{jk} \right) f_k \right)$$

Letting

$$dA = (dA_{ik})_{1 \le i,k \le n} \quad \text{and} \quad A \wedge A = \left(\sum_{j=1}^{n} A_{ij} \wedge A_{jk} \right)_{1 \le i,k \le n}$$

which are $n \times n$ matrices with entries in Ω_X^2

then

$$\nabla^2 \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} = (dA + A \wedge A) \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$ in the basis $e_1, \ldots, e_n$ of M.

The <u>curvature</u> of ∇ is the matrix

$$F = dA + A \wedge A, \quad \text{where}$$

$A = (A_{ij})$ is given by

$$\nabla(e_i) = e_1 \otimes A_{1i} + \cdots + e_n A_{ni}.$$

We have proved,

∇ is flat if and only if $F=0$.

Remarks around assignments

(a) From Quentin Bell:

"An important skill, especially in research, is asking questions. Your assignments are a form of research. As such they should have many questions (Why this? How is this useful? Is this similar to that?) While assignments should be drafted and revised before submission, including these questions will allow you to get good feedback on this aspect of research." He has said it perfectly.

(b) There are multiple (two) standardly used topologies on $\mathbb{P}^n$ used in algebraic geometry (denoted $\mathbb{P}^n$ and $\hat{\mathbb{P}}^n$ in the week 2 lecture notes). An important part of the understanding of $\mathbb{P}^n$ in algebraic geometry (and for the assignment) is to compare and contrast these topologies and determine which properties of $\mathbb{P}^n$ hold in each case.

(c) The definition (and topology) of S^2 is in the lecture notes Week 2 (additional notes and main notes). If you wish to use a different topology on S^2 that is fine, but give a careful definition, explain why you choose this alternate topology, how it relates to the topology used in class and how its relation to the two topologies on $\mathbb{P}^1$ is affected by this alternate choice.

(d) As evidenced by the heading "Projective space as a ringed space" in the lecture notes for Week 2, it is vital, in algebraic geometry not only to view projective space as a topological space, but, in many contexts, as a topological space with additional structure (an object in the category of ringed spaces). It is part of the subject to learn in which contexts this extra structure is an important part of the statements being made and in which contexts it is not.

(e) Proper referencing in mathematical writing is important to ensure that the writing contains no plagiarism (even unintentionally). Referencing, and what the goals and the culture are was discussed at some length in class. In many cases, during the marking of assignments, feedback is given on how to be more attentive, more accurate and more informative with referencing techniques and styles.

(f) Affine ~~varieties~~ space is defined in Harder, Definition 6.1.10.

(g) An important component of research training is the development of skills for using resources to obtain information. The definition of perfectoid space from Scholze's answer on MathOverflow or from Scholze's PhD thesis, provided it is given with a proper reference to the source, constituted a good answer to one part of one question on assignment 1. This definition is "A perfectoid space is a ringed space which is locally isomorphic to an affinoid perfectoid space". This definition has a very strong similarity to definitions of varieties, manifolds and schemes as given in class and in the lecture notes. This makes the definition of perfectoid space fit squarely into the topic and material covered in lectures. As explained in lecture and in the lecture notes, it is the term "locally isomorphic" which has a particular need for further amplification in the context of the material for this subject.

(h) There are headings "Affine space as a ringed space" and "Projective space as a ringed space" which strongly indicate that viewing these objects in the category of sets is not in the spirit of this course.

(i) An important goal of the course is to teach, learn and improve the quality of one's mathematical writing and mathematical exposition. The final marks for assignments (and exams) are reflective of an assessment of the exposition, the clarity, the correctness and the depth of the exploration. This mechanism for assessment is similar to assessment of an essay, a report, or a mathematics paper. The assignments do provide additional structure, as a bridge between lower level mathematics assigments and mathematical papers, but are intended to begin the process of learning a more advanced approach to mathematical exposition and thoughtful analysis.