Continuous functions are for comparing topological spaces.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A continuous function from X to Y is a function $f: X \to Y$ such that

$$\text{if } V \in \mathcal{T}_Y \text{ then } f^{-1}(V) \in \mathcal{T}_X.$$

Recall:

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}.$$

Uniformly continuous functions are for comparing uniform spaces.

Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces.

A uniformly continuous function from X to Y is a function $f: X \to Y$ such that

$$\text{if } E \in \mathcal{X}_Y \text{ then } (f \times f)^{-1}(E) \in \mathcal{X}_X.$$

Recall:

$$(f \times f)^{-1}(E) = \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in E\}.$$

Neighborhoods

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$.
The <u>neighborhood filter</u> of X is

$$\mathcal{N}(x) = \left\{ N \subseteq X \;\middle|\; \begin{array}{l} \text{there exists } U \in \mathcal{T} \text{ such that} \\ x \in U \text{ and } U \subseteq N \end{array} \right\}$$

A <u>neighborhood of x</u> is a set $N \subseteq X$ such that

$N \in \mathcal{N}(x)$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be ~~cont~~ topological spaces.
Let $f : X \to Y$. Let $a \in X$.
The function $f : X \to Y$ is <u>continuous at a</u>
if f satisfies:

if $P \in \mathcal{N}(f(a))$ then $f^{-1}(P) \in \mathcal{N}(a)$.

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f : X \to Y$ be a function. Then $f : X \to Y$ is continuous if and only if f satisfies:

if $a \in X$ then $f : X \to Y$ is continuous at a.

<u>Uniform spaces can be made into topological spaces</u>

Let $(X, \mathcal{X})$ be a uniform space

Let $E \in \mathcal{X}$ and let $x \in X$.

The <u>E-neighborhood of x</u> is

$$B_E(x) = \{ y \in X \mid (x,y) \in E \}.$$

The <u>uniform space topology on X</u> is

(*) $$\mathcal{T} = \left\{ U \subseteq X \;\middle|\; \begin{array}{l} \text{if } x \in U \text{ then there exists } E \in \mathcal{X} \\ \text{such that } B_E(x) \subseteq U \end{array} \right\}$$

<u>Theorem</u> Let $(X, \mathcal{X})$ be a uniform space and let $\mathcal{T}$ be as defined in (*). Then $(X, \mathcal{T})$ is a topological space.

<u>Uniformly continuous functions are continuous</u>

Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces.
Let $\mathcal{T}_X$ be the uniform space topology on X.
Let $\mathcal{T}_Y$ be the uniform space topology on Y
Let $f: X \to Y$ be a uniformly continuous function
To show: $f: X \to Y$ is continuous.

Proof Assume $f: X \to Y$ is uniformly continuous.

To show: $f: X \to Y$ is continuous.

To show: If $a \in X$ then f is continuous at a.

Assume $a \in X$.

To show: f is continuous at a.

To show: If $P \in \mathcal{N}(f(a))$ then $f^{-1}(P) \in \mathcal{N}(a)$.

Assume $P \in \mathcal{N}(f(a))$.

To show: $f^{-1}(P) \in \mathcal{N}(a)$.

To show: There exists $D \in \mathcal{X}_X$ such that

$$f^{-1}(P) \supseteq B_D(a)$$

Since $P \in \mathcal{N}(f(a))$ there exists $E \in \mathcal{X}_Y$ such that $P \supseteq B_E(f(a))$

Let $D = (f \times f)^{-1}(E)$.

To show: $f^{-1}(P) \supseteq B_D(a)$

To show: If $x \in B_D(a)$ then $x \in f^{-1}(P)$.

Assume $x \in B_D(a)$.

Then $(a,x) \in D$

So $(a,x) \in (f \times f)^{-1}(E)$

So $(f(a), f(x)) \in E$.

So $f(x) \in B_\varepsilon(f(a)) \subseteq P$

So $x \in f^{-1}(P)$.

So $f^{-1}(P) \supseteq B_\delta(a)$

So $f^{-1}(P) \in \mathcal{N}(a)$

So f is continuous at a.

So f is continuous.