06.10.2017
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Metric and Hilbert Spaces: Lecture 30

Completions

Let (X,d) be a metric space.

The completion of (X,d) is a metric space $(\hat{X},\hat{d})$ with an isometry $\iota: X \to \hat{X}$ such that

(a) $(\hat{X},\hat{d})$ is complete,

(b) $\overline{\iota(X)} = \hat{X}$.

Isometries: Let (X,d_X) and (Y,d_Y) be metric spaces. An isometry from X to Y is a function

$\varphi: X \to Y$ such that

if $x_1, x_2 \in X$ then $d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1,x_2)$.

Recall that isometries are injective.

Existence of the completion

The completion of (X,d) is

$$\hat{X} = \{\text{Cauchy sequences in } X\}$$ with

$\hat{d}: \hat{X}\times\hat{X} \to \mathbb{R}_{\geq 0}$ given by

$$\hat{d}(\vec{x},\vec{y}) = \lim_{n\to\infty} d(x_n,y_n)$$

and with $\iota: X \to \hat{X}$

$$x \mapsto (x,x,x,\ldots)$$

and Cauchy sequences $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ are <u>equal in $\hat{X}$</u>,

$$\vec{x} = \vec{y} \quad \text{if} \quad \lim_{n\to\infty} d(x_n, y_n) = 0.$$

<u>Uniqueness of the completion</u>

Let (X,d) be a metric space.
Let $(\hat{X}_1, \hat{d}_1, \hat{\imath}_1)$ and $(\hat{X}_2, \hat{d}_2, \hat{\imath}_2)$ be completions of (X,d). Then there exists a bijective isometry

$$\hat{X}_1 \xrightarrow{\varphi} \hat{X}_2$$

<u>Proof</u> To show: There exists $\varphi: \hat{X}_1 \to \hat{X}_2$ such that φ is a bijective isometry.

Let $z \in \hat{X}_1$.

Since $\overline{\imath_1(X)} = \hat{X}_1$ then there exists a sequence $(\imath_1(x_1), \imath_1(x_2), \ldots)$ in $\imath_1(X)$ with $\lim_{n\to\infty} \imath_1(x_n) = z$

Define

$$\varphi(z) = \varphi\left(\lim_{n\to\infty} \imath_1(x_n)\right) = \lim_{n\to\infty} \varphi(\imath_1(x_n))$$
$$= \lim_{n\to\infty} \imath_2 \imath_1^{-1} \imath_1(x_n) = \lim_{n\to\infty} \imath_2(x_n).$$

To show: (a) φ is an isometry
(b) $\varphi: \hat{X}_1 \to \hat{X}_2$ is surjective.

(b) To show: If $z \in \hat{X}_2$ then there exists $w \in \hat{X}_1$ such that $\varphi(w) = z$.

Assume $z \in \hat{X}_2$

To show: There exists $w \in \hat{X}_1$ with $\varphi(w) = z$

Since $z \in \hat{X}_2 = \overline{\imath_2(X)}$ then there exists $(\imath_2(x_1), \imath_2(x_2), \ldots)$ with $\lim_{n\to\infty} \imath_2(x_n) = z$

Since $(\imath_2(x_1), \imath_2(x_2), \ldots)$ converges then $(\imath_2(x_1), \imath_2(x_2), \ldots)$ is Cauchy.

Since $\imath_2$ is an isometry then $(x_1, x_2, \ldots)$ is Cauchy.

Since $\imath_1$ is an isometry then $(\imath_1(x_1), \imath_1(x_2), \ldots)$ is Cauchy.

Since $\hat{X}_1$ is complete then $(\imath_1(x_1), \imath_1(x_2), \ldots)$ converges.

Let $w = \lim_{n\to\infty} \imath_1(x_n)$.

To show: $\varphi(w) = z$

$$\varphi(w) = \varphi\left(\lim_{n\to\infty} z_1(x_n)\right) = \lim_{n\to\infty} z_2(x_n) = z.$$

So φ is surjective.

<u>Examples of completions</u>

(1) Let $X = \mathbb{Q}$ with $d\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = \left|\frac{a_1}{b_1} - \frac{a_2}{b_2}\right|$.

Then $\hat{X} = \mathbb{R}$.

(2) Let $X = \mathbb{Q}$ with

$$d_p\left(\frac{a_1}{b_1}, \frac{a_2}{b_2}\right) = e^{-\mathrm{val}_p\left(\frac{a_1}{b_1} - \frac{a_2}{b_2}\right)}.$$

Then $\hat{X} = \mathbb{Q}_p$.

(3) Let $X = \mathbb{C}[t]$ with

$$d_t(f_1, f_2) = e^{-\mathrm{val}_t(f_1 - f_2)}$$

Then $\hat{X} = \mathbb{C}[[t]]$.

(4) Let $X = \mathbb{C}(t)$ with $d_t\left(\frac{f_1}{g_1}, \frac{f_2}{g_2}\right) = e^{-\mathrm{val}_t\left(\frac{f_1}{g_1} - \frac{f_2}{g_2}\right)}$

Then $\hat{X} = \mathbb{C}((t))$