Metric and Hilbert Spaces: Lecture 27

Properties of eigenspaces

Let $(H, \langle\,\rangle)$ be an inner product space and let

$T: H \to H$ be a bounded self adjoint linear operator.

The λ-eigenspace of T is

$$H_\lambda = \{v \in H \mid Tv = \lambda v\}.$$

(a) If $H_\lambda \neq 0$ then $\lambda \in \mathbb{R}$.

Proof: Let $v \in H_\lambda$ with $v \neq 0$.

Since T is self adjoint then

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Tv, v\rangle = \langle v, Tv\rangle$$
$$= \overline{\langle Tv, v\rangle} = \overline{\langle \lambda v, v\rangle} = \bar\lambda\, \overline{\langle v, v\rangle} = \bar\lambda \langle v, v\rangle$$

and $\langle v, v\rangle \neq 0$.

So $\lambda = \bar\lambda$. So $\lambda \in \mathbb{R}$.

(b) If $\lambda \neq \gamma$ then $H_\lambda \perp H_\gamma$.

Proof: Assume $\lambda \neq \gamma$.

To show: $H_\lambda \perp H_\gamma$

To show: If $x \in H_\lambda$ and $y \in H_\gamma$ then $\langle x, y\rangle = 0$.

Assume $x \in H_\lambda$ and $y \in H_\gamma$.
Since T is self adjoint then

$$\lambda\langle x,y\rangle = \langle Tx,y\rangle = \langle x,Ty\rangle = \gamma\langle x,y\rangle.$$

So $(\lambda-\gamma)\langle x,y\rangle = 0$.

Since $\lambda-\gamma \neq 0$ then $\langle x,y\rangle = 0$.

So $H_\lambda \perp H_\gamma$.

(c) Assume $T: H \to H$ is compact and $\lambda \neq 0$.
Then H_λ is finite dimensional.

__Proof__: To show: If $\lambda \neq 0$ and $T: H \to H$ compact then H_λ is finite dimensional.

Assume $\lambda \neq 0$.
To show: If $T: H \to H$ is compact then H_λ is finite dimensional.

~~Assume~~ To show: If H_λ is infinite dimensional then $T: H \to H$ is not compact.
Assume H_λ is infinite dimensional.

Let $(e_1, e_2, \ldots)$ be an orthonormal sequence in H_λ.
If $m \neq n$ then

$$\|Te_m - Te_n\| = \|\lambda e_m - \lambda e_n\| = |\lambda| \cdot \|e_m - e_n\| = |\lambda| \cdot \sqrt{2}$$

So no subsequence of $(Te_1, Te_2, \ldots)$ is Cauchy.

So no subsequence of $(Te_1, Te_2, \ldots)$ converges

So $(Te_1, Te_2, \ldots)$ does not have a cluster point.

So T is not compact.

(d) If $T: H \to H$ is a compact operator and $(\lambda_1, \lambda_2, \ldots)$ is a sequence of distinct eigenvalues then $\lim_{n\to\infty} \lambda_n = 0$

<u>Proof</u> Assume $T: H \to H$ is a linear operator and $(\lambda_1, \lambda_2, \ldots)$ is a sequence of distinct eigenvalues of T.

To show: If $T: H \to H$ is compact then

$$\lim_{n\to\infty} \lambda_n = 0.$$

To show: If $\lim_{n\to\infty} \lambda_n \neq 0$ then $T: H \to H$ is not compact.

Assume $\lim_{n\to\infty} \lambda_n \neq 0$.

To show: $T: H \to H$ is not compact.

Since $\lim_{n\to\infty} \lambda_n \neq 0$ then there exists $c \in \mathbb{R}_{>0}$ and a subsequence

$(\lambda_{k_1}, \lambda_{k_2}, \ldots)$ such that $|\lambda_{k_j}| > c$ for $j \in \mathbb{Z}_{>0}$.

Let $e_1, e_2, \ldots$ be such that $\|e_i\| = 1$ and $Te_j = \lambda_{k_j} e_j$.

Since $\lambda_{k_1}, \lambda_{k_2}, \ldots$ are all distinct then $\langle e_i, e_j \rangle = 0$.

If $m \neq n$ then

$$\|Te_m - Te_n\|^2 = \|\lambda_{k_m} e_m - \lambda_{k_n} e_n\|^2$$

$$= \langle \lambda_{k_m} e_m - \lambda_{k_n} e_n, \lambda_{k_m} e_m - \lambda_{k_n} e_n \rangle$$

$$= |\lambda_{k_m}|^2 \|e_m\|^2 + |\lambda_{k_n}|^2 \|e_n\|^2 > 2C^2.$$

So no subsequence of $(Te_1, Te_2, \ldots)$ is Cauchy.
So no subsequence of $(Te_1, Te_2, \ldots)$ converges
So $(Te_1, Te_2, \ldots)$ does not have a cluster point.
So T is not compact. //

<u>Theorem</u> Let $(H, \langle\,\rangle)$ be a Hilbert space. Let $T: H \to H$ be a bounded self adjoint compact linear operator. Let

$$W = \bigoplus_{\lambda \in \sigma_p(T)} H_\lambda,$$

where $H_\lambda = \{v \in H \mid Tv = \lambda v\}$ and $\sigma_p(T) = \{\lambda \mid H_\lambda \neq 0\}$.

Then $H = \overline{W}$