Metric and Hilbert Spaces: Lecture 19 04.09.2017 UniMelb A. Ram ①

A **Banach space** is a normed vector space which is complete (metric $d(x,y) = \|y-x\|$).

A **Hilbert space** is an inner product space $(V, \langle\, \rangle)$ which is complete (norm $\|v\| = \sqrt{\langle v, v\rangle}$).

Theorem Let $(V, \|\ \|)$ be a normed vector space. Then V is complete if and only if V satisfies:

if $(a_1, a_2, \ldots)$ is a sequence in V and $\sum_{i \in \mathbb{Z}_{>0}} \|a_i\|$ converges then $\sum_{i \in \mathbb{Z}_{>0}} a_i$ converges.

Proof $\Rightarrow$ Assume V is complete.

To show: If $\sum_{i \in \mathbb{Z}_{>0}} \|a_i\|$ converges then $\sum_{i \in \mathbb{Z}_{>0}} a_i$ converges.

Assume $(a_1, a_2, \ldots)$ is a sequence in V and $\sum_{i \in \mathbb{Z}_{>0}} \|a_i\|$ converges.

To show: $\sum_{i \in \mathbb{Z}_{>0}} a_i$ converges.

Let
$$s_n = \sum_{i=1}^{n} a_n \quad \text{and} \quad S_n = \sum_{i=1}^{n} \|a_i\|.$$

Since $(S_1, S_2, \ldots)$ converges then $(S_1, S_2, \ldots)$ is Cauchy.

Since
$$\|s_n - s_m\| = \Big\|\sum_{i=n}^{m} a_i\Big\| \le \sum_{i=n}^{m} \|a_i\| = \|S_n - S_m\|$$
the sequence $(s_1, s_2, \ldots)$ is Cauchy.

Since V is complete then $(s_1, s_2, \ldots)$ converges.

So $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

$\Leftarrow$ Assume that V satisfies $(*)$.

To show: V is complete.

To show: If $(s_1, s_2, \ldots)$ is a Cauchy sequence in V then $(s_1, s_2, \ldots)$ converges.

Assume $(s_1, s_2, \ldots)$ is a Cauchy sequence in V.

Let $k_n \in \mathbb{Z}_{>0}$ be such that if $r, m \in \mathbb{Z}_{\ge k_n}$
then $\|s_r - s_m\| \le \frac{1}{10^n}$

Let
$$a_1 = s_{k_1},\ a_2 = s_{k_1} - s_{k_2},\ a_3 = s_{k_3} - s_{k_2}, \ldots$$

Then
$$\|a_n\| \le \frac{1}{10^n}$$

So
$$\sum_{n\in\mathbb{Z}_{>0}} \|a_n\| \le \sum_{n\in\mathbb{Z}_{>0}} \frac{1}{10^n} = \frac{1}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\right) = \frac{1}{10}\,\frac{1}{1 - \frac{1}{10}}$$
$$= \frac{1}{9}.$$

So $\sum_{n\in\mathbb{Z}_{>0}} \|a_n\|$ converges.

Since V satisfies (*) then $\sum_{n\in\mathbb{Z}_{>0}} a_n$ converges.

So $(s_{k_1}, s_{k_2}, \ldots)$ converges since

$$s_{k_1}=a_1,\quad s_{k_2}=a_1+a_2,\quad s_{k_3}=a_1+a_2+a_3, \ldots$$

So $(s_1, s_2, \ldots)$ has a cluster point.

Since $(s_1, s_2, \ldots)$ is Cauchy and has a cluster point then $(s_1, s_2, \ldots)$ converges.

So V is complete. //

<u>Theorem</u> Let $(V, \|\ \|)$ and $(W, \|\ \|)$ be normed vector spaces. If W is complete then $B(V,W)$ is complete.

<u>Proof</u> Assume W is complete.

To show: If $T_1, T_2, \ldots$ is a Cauchy sequence in $B(V,W)$ then $T_1, T_2, \ldots$ converges in $B(V,W)$.

Assume $T_1: V\to W$, $T_2: V\to W$, $\ldots$ is a Cauchy sequence in $B(V,W)$.

To show: There exists $T: V\to W$ with $T\in B(V,W)$ such that $\lim_{n\to\infty} T_n = T$.

Define $T: V \to W$ by

$$T(x) = \lim_{n\to\infty} T_n(x).$$

To show: (a) T is a function ($T(x)$ exists)

(b) $T \in B(V, W)$

(c) $\lim_{n\to\infty} T_n = T$.

(c) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then

$$\|T - T_n\| < \varepsilon.$$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: there exists $N \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq N}$ then $\|T - T_n\| < \varepsilon$.

Using that $(T_1, T_2, \ldots)$ is Cauchy, let $N \in \mathbb{Z}_{>0}$ such that if $m, n \in \mathbb{Z}_{\geq N}$ then

$$\|T_m - T_n\| < \frac{\varepsilon}{2}.$$

To show: If $n \in \mathbb{Z}_{\geq N}$ then $\|T - T_n\| < \varepsilon$

Assume $n \in \mathbb{Z}_{\geq N}$.

To show: $\|T - T_n\| < \varepsilon$.

To show: $\sup\left\{ \frac{\|(T - T_n)(x)\|}{\|x\|} \;\middle|\; x \in V \text{ and } x \neq 0 \right\} < \varepsilon$.

Assume $x \in V$ and $x \neq 0$.

To show: $\dfrac{\|(T-T_n)(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$

To show: $\dfrac{\|T(x)-T_n(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$

To show: $\dfrac{\|\lim_{m\to\infty} T_m(x) - T_n(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$

Using that $\| \ \| : W \to \mathbb{R}_{\geq 0}$ is continuous,

To show: $\lim_{m\to\infty} \dfrac{\|T_m(x)-T_n(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$.

To show: There exists $M \in \mathbb{Z}_{>0}$ such that if $m \in \mathbb{Z}_{\geq M}$ then $\dfrac{\|T_m(x)-T_n(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$

Let $M = N$.

To show: If $m \in \mathbb{Z}_{\geq N}$ then $\dfrac{\|T_m(x)-T_n(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$.

Assume $m \in \mathbb{Z}_{\geq N}$.

To show: $\dfrac{\|T_m(x)-T_n(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$.

Since $m, n \in \mathbb{Z}_{\geq N}$ then

$$\frac{\varepsilon}{2} > \|T_m - T_n\|$$

$$= \sup\left\{ \frac{\|T_m(y) - T_n(y)\|}{\|y\|} \;\middle|\; y \in V \text{ and } y \neq 0 \right\}$$

$$\geq \frac{\|T_m(x) - T_n(x)\|}{\|x\|}$$

So $\dfrac{\|T_m(x) - T_n(x)\|}{\|x\|} < \dfrac{\varepsilon}{2}$

So $\sup\left\{ \dfrac{\|T(x) - T_n(x)\|}{\|x\|} \;\middle|\; x \in V \text{ and } x \neq 0 \right\} \leq \dfrac{\varepsilon}{2}$

So $\|T - T_n\| < \varepsilon$

So $\lim\limits_{n \to \infty} T_n = T.$ //.

~~So $\|T - T_n\|$~~