Metric and Hilbert spaces: Lecture 13

Limits and closure

Theorem Let (X,d) be a metric space. Let $A \subseteq X$.

Then

$$\bar{A} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence } (a_1, a_2, \ldots) \\ \text{in } A \text{ with } z = \lim_{n\to\infty} a_n \end{array} \right\}$$

Recall: $z = \lim_{n\to\infty} a_n$ means:

If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $d(a_n, z) < \varepsilon$.

i.e. If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in B_\varepsilon(z)$.

A neighborhood of z is $N \subseteq X$ such that there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(z) \subseteq N$.

So $z = \lim_{n\to\infty} a_n$ is equivalent to
if $N \in \mathcal{N}(z)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.

Proof of the theorem Let

$$R = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence } (a_1, a_2, \ldots) \\ \text{in } A \text{ with } z = \lim_{n\to\infty} a_n \end{array} \right\}$$

To show: $\bar{A} = R$.

To show: (a) $R \subseteq \bar{A}$

(b) $\bar{A} \subseteq R$.

(a) To show: If $z \in R$ then $z \in \bar{A}$.

Assume $z \in R$

To show: $z \in \bar{A}$.

Since $z \in R$ then there exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n \to \infty} a_n$

To show: z is a close point to A.

To show: If $N \in \mathcal{N}(z)$ then $N \cap A \neq \emptyset$

Assume $N \in \mathcal{N}(z)$.

Since $\lim_{n \to \infty} a_n = z$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.

So $N \cap A \neq \emptyset$.

So z is a close point of A.

So $R \subseteq \bar{A}$.

(b) To show: $\bar{A} \subseteq R$.

To show: If $z \in \bar{A}$ then $z \in R$.

Assume $z \in \bar{A}$

To show: $z \in R$.

To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n \to \infty} a_n$.

Using that z is a close point to A, let

$$a_1 \in B_{10^{-1}}(z) \cap A, \quad a_2 \in B_{10^{-2}}(z) \cap A, \ \ldots$$

To show: $z = \lim_{n\to\infty} a_n$.

To show: If N is a neighborhood of z then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.

Assume $N \in \mathcal{N}(z)$.

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.

Let $\ell \in \mathbb{Z}_{>0}$ such that $B_{10^{-\ell}}(z) \subseteq N$.

To show: If $n \in \mathbb{Z}_{\geq \ell}$ then $a_n \in N$.

Assume $n \in \mathbb{Z}_{\geq \ell}$.

To show: $a_n \in N$.

Since $n \geq \ell$ then $10^{-n} \leq 10^{-\ell}$ and

$$a_n \in B_{10^{-n}}(z) \subseteq B_{10^{-\ell}}(z) \subseteq N.$$

So $z = \lim_{n\to\infty} a_n$.

So $z \in R$.

So $\overline{A} \subseteq R$.

So $\overline{A} = R$. //

A <u>Hausdorff topological space</u> is a topological space $(X, \mathcal{T})$ such that
if $x, y \in X$ and $x \neq y$ then there exist
$U \in \mathcal{N}(x)$ and $V \in \mathcal{N}(y)$ such that $U \cap V = \emptyset$.

<u>Proposition</u> Let $(X, \mathcal{T})$ be a topological space. Assume $(X, \mathcal{T})$ is Hausdorff. Let $(x_1, x_2, \ldots)$ be a sequence in X.
If $\lim_{n \to \infty} x_n$ exists in X then $\lim_{n \to \infty} x_n$ is unique.

<u>Proof</u>

Assume $z_1 = \lim_{n \to \infty} x_n$ and $z_2 = \lim_{n \to \infty} x_n$

To show: $z_1 = z_2$.
Proof by contradiction.
Assume $z_1 \neq z_2$.
Using that X is Hausdorff let
$N_1 \in \mathcal{N}(z_1)$ and $N_2 \in \mathcal{N}(z_2)$ with $N_1 \cap N_2 = \emptyset$.
Since $\lim_{n \to \infty} x_n = z_1$ there exists $\ell_1 \in \mathbb{Z}_{>0}$ such that
$\{x_{\ell_1}, x_{\ell_1 + 1}, \ldots\} \subseteq N_1$.
Since $\lim_{n \to \infty} x_n = z_2$ there exists $\ell_2 \in \mathbb{Z}_{>0}$ such that
$\{x_{\ell_2}, x_{\ell_2 + 1}, \ldots\} \subseteq N_2$.

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Let $l = \max\{l_1, l_2\}$.

Then $\{x_l, x_{l+1}, \ldots\} \subseteq N_1 \cap N_2$.

This is a contradiction to $N_1 \cap N_2 = \emptyset$.

So $z_1 = z_2$. //