(7a) Let V be an inner product space

Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in V.

To show: $\{a_1, a_2, \ldots\}$ is linearly independent.

To show: If $n \in \mathbb{Z}_{>0}$ and $c_1, \ldots, c_n \in \mathbb{K}$ ~~then~~ and $c_1 a_1 + \cdots + c_n a_n = 0$ then $c_1 = 0, c_2 = 0, \ldots, c_n = 0$.

Assume $n \in \mathbb{Z}_{>0}$ and $c_1, \ldots, c_n \in \mathbb{K}$ and $c_1 a_1 + \cdots + c_n a_n = 0$.

To show: If $j \in \{1, \ldots, n\}$ then $c_j = 0$.

Assume $j \in \{1, 2, \ldots, n\}$.

To show: $c_j = 0$.

$$c_j = \langle c_1 a_1 + \cdots + c_n a_n, a_j \rangle = \langle 0, a_j \rangle = 0,$$

since $\langle a_i, a_j \rangle = \cancel{0, \ldots, a_n}\, \delta_{ij}$.

So $c_j = 0$.

So $\{a_1, a_2, \ldots\}$ is linearly independent.

(7b) Let us copy this from Prop 1.3.4.1 on p. 54 of part III of the notes.

To show: If $a_1 = \frac{v_1}{\|v_1\|}$ and

$$a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n-1}, a_n\rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n-1}, a_n\rangle a_n\|}$$

then $(a_1, a_2, \ldots)$ is an orthonormal sequence.

<u>Proof</u> by induction on n.

Let $n \in \mathbb{Z}_{>0}$.

Assume that if $k, m \in \mathbb{Z}_{<n}$ then $\langle a_k, a_m\rangle = \delta_{km}$.

To show: If $m, n \in \mathbb{Z}_{\geq 0}$ and $m \leq n$ then $\langle a_m, a_n\rangle = \delta_{mn}$

Assume $m, n \in \mathbb{Z}_{\geq 0}$ and $m \leq n$.

If $m < n$ then

$$\langle a_m, a_n\rangle = \frac{\langle a_m, v_n - \langle v_n, a_1\rangle a_1 - \cdots - \langle v_n, a_{n-1}\rangle a_{n-1}\rangle}{\|v_n - \langle v_n, a_1\rangle a_1 - \cdots - \langle v_n, a_{n-1}\rangle a_{n-1}\|}$$

$$= \frac{\langle a_m, v_n\rangle - \langle v_n, a_m\rangle \langle a_m, a_m\rangle}{\|v_n - \langle v_n, a_1\rangle a_1 - \cdots - \langle v_n, a_{n-1}\rangle a_{n-1}\|} = 0.$$

and, if $m = n$ then $\langle a_m, a_n\rangle$ equals

$$\frac{\langle v_n - \langle v_n, a_1\rangle a_1 - \cdots - \langle v_n, a_{n-1}\rangle a_{n-1}, v_n - \langle v_n, a_1\rangle a_1 - \cdots - \langle v_n, a_{n-1}\rangle a_{n-1}\rangle}{\|v_n - \langle v_n, a_1\rangle a_1 - \cdots - \langle v_n, a_{n-1}\rangle a_{n-1}\|^2}$$

$= 1.$

So $\langle a_m, a_n \rangle = \delta_{mn}$.

So $(a_1, a_2, \ldots)$ is an orthonormal sequence in V.

(c) Let

$$d_1 = \|v_1\|$$
$$d_2 = \|v_2 - \langle v_2, a_1 \rangle a_1\|$$
$$d_3 = \|v_3 - \langle v_3, a_1 \rangle a_1 - \langle v_3, a_2 \rangle a_2\|$$

and so on, so that

$$d_n = \|v_n - \langle v_n, a_1 \rangle a_1 - \langle v_n, a_2 \rangle a_2 - \cdots - \langle v_n, a_{n-1} \rangle a_{n-1}\|.$$

Let P be the change of basis matrix from $(v_1, v_2, \ldots, v_n)$ to $(a_1, a_2, \ldots, a_n)$ so that

$$a_i = \sum_{l=1}^{n} P_{il} v_l .$$

Then
$$\delta_{ij} = \langle a_i, a_j \rangle = \left\langle \sum_{l=1}^{n} P_{il} v_l, \sum_{k=1}^{n} P_{jk} v_k \right\rangle$$
$$= \sum_{l=1}^{n} \sum_{k=1}^{n} P_{il} \langle v_l, v_k \rangle \overline{P}_{jk}$$
$$= (PA\overline{P}^t)_{ij} .$$

So $PA\overline{P}^t = I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$.

$P_1 = \begin{pmatrix} \frac{1}{d_1} & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix}$ is the change of basis matrix from $(v_1, v_2, \dots, v_n)$ to $(a_1, v_2, \dots v_n)$

$P_2 = \frac{1}{d_2} \begin{pmatrix} 1 & 0 & & \\ -\langle v_2, a_1 \rangle & 1 & & 0 \\ & & 1 & \\ & 0 & & \ddots \; 1 \end{pmatrix}$ is the change of basis matrix from $(a_1, v_2, v_3, \dots, v_n)$ to $(a_1, a_2, v_3, \dots, v_n)$.

$P_3 = \frac{1}{d_3} \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ -\langle v_3, a_1 \rangle & -\langle v_3, a_2 \rangle & 1 & & 0 \\ & & & 1 & \\ & 0 & & & \ddots \; 1 \end{pmatrix}$

is the change of basis matrix from $(a_1, a_2, v_3, \dots, v_n)$ to $(a_1, a_2, a_3, v_4, \dots v_n)$.

So $P = P_n P_{n-1} \cdots P_2 P_1$ and $\det(P_k) = \frac{1}{d_k}$.

So $1 = \det(PA\bar{P}^t) = \det(P_n \cdots P_2 P_1 A \bar{P}_1^t \bar{P}_2^t \cdots \bar{P}_n^t)$

$= \det(P_n) \cdots \det(P_1) \det(A) \det(\bar{P}_1^t) \cdots \det(\bar{P}_n^t)$

$= \frac{1}{d_n} \cdots \frac{1}{d_2} \cdot \frac{1}{d_1} \det(A) \frac{1}{\bar{d}_n} \frac{1}{\bar{d}_{n-1}} \cdots \frac{1}{\bar{d}_2} \frac{1}{\bar{d}_1}$

$= \frac{1}{|d_n|^2 \cdots |d_2|^2 |d_1|^2} \cdot \det(A).$

So $\det(A) = |d_1|^2 |d_2|^2 \cdots |d_n|^2$

$= d_1^2 d_2^2 \cdots d_n^2$ since $d_k \in \mathbb{R}_{>0}$.

Note that the question claims

$\det(A) = d_n$,

this proof corrects the question and shows

$\det(A) = d_1^2 d_2^2 \cdots d_n^2$.