(5a) Following the Lecture notes Part III §1.30, page 49.

Theorem Let (X,d) be a complete metric space. Let $U_1, U_2, \ldots$ be open dense subsets of X. Then

$$U_1 \cap U_2 \cap \cdots \quad \text{is dense in } X.$$

Proof To show: If (X,d) is a complete metric space and $U_1, U_2, \ldots$ are open dense subsets then $U_1 \cap U_2 \cap \cdots$ is dense in X.

Assume (X,d) is a complete metric space and $U_1, U_2, \ldots$ are open dense subsets.

To show: $U_1 \cap U_2 \cap \cdots$ is dense in X.

To show: $\overline{(U_1 \cap U_2 \cap \cdots)} = X$.

To show: If $x \in X$ then $x \in \overline{(U_1 \cap U_2 \cap \cdots)}$

Assume $x \in X$.

To show: $x \in \overline{(U_1 \cap U_2 \cap \cdots)}$

To show: x is a close point to $U_1 \cap U_2 \cap \cdots$:

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $B_\varepsilon(x) \cap (U_1 \cap U_2 \cap \cdots) \neq \emptyset$

Assume $\varepsilon \in \mathbb{R}_{>0}$.

Define a sequence $(x_0, x_1, \ldots)$ in X and $(\varepsilon_0, \varepsilon_1, \ldots)$ in $\mathbb{R}_{>0}$ as follows:
Let $x_0 = x$ and $\varepsilon_0 = \varepsilon$.

Using that U_n is open and dense,
let $x_k \in X$ and $\varepsilon_{k+1} \in \mathbb{R}_{>0}$ be such that

$$\varepsilon_{k+1} < \frac{\varepsilon_k}{3} \quad \text{and} \quad B_{3\varepsilon_{k+1}}(x_{k+1}) \subseteq B_{\varepsilon_k}(x_k) \cap U_{k+1}.$$

Let $y = \lim_{n\to\infty} x_n$.

To show: (a) y exists in X

(b) $y \in B_\varepsilon(x) \cap (U_1 \cap U_2 \cap \ldots)$

(a) To show: $y = \lim_{n\to\infty} x_n$ exists in X.

To show: $(x_1, x_2, \ldots)$ converges.
Using that X is complete, ~~it suffices~~ to show that $(x_1, x_2, \ldots)$ is Cauchy.
To show: If $\delta \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>\ell}$ then $d(x_r, x_s) < \delta$

Assume $\delta \in \mathbb{R}_{>0}$.
Let $\ell \in \mathbb{Z}_{>0}$ be such that

$$\frac{1}{3^\ell} \cdot \frac{3}{2} \cdot \varepsilon < \delta.$$

To show: If $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_r, x_s) < \gamma$.

Assume $r, s \in \mathbb{Z}_{\geq \ell}$.

To show: $d(x_r, x_s) < \gamma$.

Using the triangle inequality,

$$d(x_r, x_s) \leq d(x_r, x_{r+1}) + d(x_{r+1}, x_{r+2}) + \cdots + d(x_{s-1}, x_s)$$
$$< \varepsilon_r + \varepsilon_{r+1} + \cdots + \varepsilon_{s-1}.$$

Since $\varepsilon_{k+1} < \frac{\varepsilon_k}{3}$ for $k \in \mathbb{Z}_{>0}$ then $\varepsilon_{k+n} < \frac{1}{3^n}\varepsilon_k$ for $n \in \mathbb{Z}_{>0}$.

So

$$d(x_r, x_s) < \varepsilon_r + \varepsilon_{r+1} + \cdots + \varepsilon_{s-1} < \left(\frac{1}{3^r} + \frac{1}{3^{r+1}} + \cdots + \frac{1}{3^s}\right)\varepsilon$$
$$< \frac{1}{3^r}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)\varepsilon = \frac{1}{3^r}\left(\frac{1}{1-\frac{1}{3}}\right)\varepsilon$$
$$= \frac{1}{3^r} \cdot \frac{3}{2} \cdot \varepsilon \leq \frac{1}{3^\ell} \cdot \frac{3}{2} \cdot \varepsilon < \gamma.$$

So $(x_1, x_2, \ldots)$ is Cauchy.

So $(x_1, x_2, \ldots)$ converges.

So $y = \lim_{n \to \infty} x_n$ exists

(5b) To show: $y \in B_\varepsilon(x) \cap (U_1 \cap U_2 \cap \cdots)$

Using that

$$B_{3\varepsilon_k}(x_k) \subseteq B_{\varepsilon_{k-1}}(x_{k-1}) \cap U_k \subseteq U_k$$

To show: $y \in B_\varepsilon(x) \cap B_{3\varepsilon_1}(x) \cap B_{3\varepsilon_2}(x) \cap \cdots$

To show: If $k \in \mathbb{Z}_{>0}$ then $d(y, x_k) < 3\varepsilon_k$

Assume $k \in \mathbb{Z}_{>0}$.

To show: $d(y, x_k) < 3\varepsilon_k$.

$$d(y, x_k) = d(\lim_{n\to\infty} x_n, x_k)$$

$$= \lim_{n\to\infty} d(x_n, x_k), \quad \text{since } d(\cdot, x_k): X \to \mathbb{R}_{\geq 0} \text{ is continuous,}$$

$$\leq \lim_{n\to\infty} \left(d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{n-1}, x_n)\right)$$

$$\leq \lim_{n\to\infty} \left(\varepsilon_k + \frac{1}{3}\varepsilon_k + \frac{1}{3^2}\varepsilon_k + \cdots + \frac{1}{3^{n-k}}\varepsilon_k\right)$$

$$= \varepsilon_k \lim_{n\to\infty} \left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-k}}\right)$$

$$= \varepsilon_k \sum_{j=0}^{\infty} \frac{1}{3^j} = \varepsilon_k \frac{1}{1-\frac{1}{3}} = \varepsilon_k \cdot \frac{3}{2} < 3\varepsilon_k.$$

So $y \in B_\varepsilon(x) \cap B_{3\varepsilon_1}(x) \cap B_{3\varepsilon_2}(x) \cap \cdots$

So $y \in B_\varepsilon(x) \cap (U_1 \cap U_2 \cap \cdots)$

So $B_\varepsilon(x) \cap (U_1 \cap U_2 \cap \cdots) \neq \phi$

So x is a close point to $(U_1 \cap U_2 \cap \cdots)$.

So $\overline{(U_1 \cap U_2 \cap \cdots)} = X$.

So $U_1 \cap U_2 \cap \cdots$ is dense in X.