(3a) Let X be a set and $d: X\times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ a pseudometric. Let

$$\mathcal{X} = \left\{ V \subseteq X\times X \;\middle|\; \begin{array}{l}\text{there exists } \varepsilon \in \mathbb{R}_{>0} \text{ such that} \\ V \supseteq B_\varepsilon\end{array} \right\}$$

To show: $\mathcal{X}$ is a uniformity.

To show: (aa) If $V \in \mathcal{X}$ then $V \supseteq \Delta(X)$

(ab) If $V \in \mathcal{X}$ and $D \subseteq X\times X$ and $D \supseteq V$ then $D \in \mathcal{X}$

(ac) If $V \in \mathcal{X}$ then $\sigma(V) \in \mathcal{X}$.

(ad) If $V_1, V_2, \ldots, V_\ell \in \mathcal{X}$ then $V_1 \cap V_2 \cap \cdots \cap V_\ell \in \mathcal{X}$

(ae) If $V \in \mathcal{X}$ then there exists $D \in \mathcal{X}$ such that $D \times_X D \subseteq V$.

(aa) Assume $V \in \mathcal{X}$

To show: $V \supseteq \Delta(X)$.

$$B_\varepsilon = \{(x,y) \in X\times X \mid d(x,y) < \varepsilon\}$$
$$\supseteq \{(x,y) \in X\times X \mid d(x,y) = 0\} = \Delta(X).$$

Since there exists $\varepsilon \in \mathbb{R}_{>0}$ with $V \supseteq B_\varepsilon$ then

$$V \supseteq B_\varepsilon \supseteq \Delta(X).$$

(ab) Assume $V \in \mathcal{X}$ and $D \subseteq X \times X$ and $D \supseteq V$.

To show: $D \in \mathcal{X}$.

Since $V \in \mathcal{X}$ there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $V \supseteq B_\varepsilon$.

So $D \supseteq V \supseteq B_\varepsilon$.

So there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $D \supseteq B_\varepsilon$.

So $D \in \mathcal{X}$.

(ac) Assume $V \in \mathcal{X}$.

To show: $\sigma(V) \in \mathcal{X}$.

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $V \supseteq B_\varepsilon$.

To show: $\sigma(V) \supseteq B_\varepsilon$.

$$\begin{aligned}\sigma(V) &= \{(y,x) \mid (x,y) \in V\} \\ &\supseteq \{(y,x) \mid (x,y) \in B_\varepsilon\} \\ &= \{(y,x) \mid d(x,y) < \varepsilon\} = B_\varepsilon.\end{aligned}$$

So there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $\sigma(V) \supseteq B_\varepsilon$.

So $\sigma(V) \in \mathcal{X}$.

(ad) Assume $\ell \in \mathbb{Z}_{>0}$ and $V_1, V_2, \ldots, V_\ell \in \mathcal{X}$.

To show: $V_1 \cap V_2 \cap \cdots \cap V_\ell \in \mathcal{X}$.

To show: There exists $\varepsilon \in \mathbb{R}_{>0}$ such that

$$V_1 \cap V_2 \cap \cdots \cap V_\ell \supseteq B_\varepsilon.$$

Let $\varepsilon_1 \in \mathbb{R}_{>0}$ such that $V_1 \supseteq B_{\varepsilon_1}$,

$\varepsilon_2 \in \mathbb{R}_{>0}$ such that $V_2 \supseteq B_{\varepsilon_2}$, …,

$\varepsilon_\ell \in \mathbb{R}_{>0}$ such that $V_\ell \supseteq B_{\varepsilon_\ell}$

Let $\varepsilon \in \mathbb{R}_{>0}$ be $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_\ell\}$.

To show: $V_1 \cap V_2 \cap \cdots \cap V_\ell \supseteq B_\varepsilon$

$$V_1 \cap V_2 \cap \cdots \cap V_\ell \supseteq B_{\varepsilon_1} \cap B_{\varepsilon_2} \cap \cdots \cap B_{\varepsilon_\ell} = B_\varepsilon.$$

So $V_1 \cap V_2 \cap \cdots \cap V_\ell \in \mathcal{X}$.

(ae) Assume $V \in \mathcal{X}$.

To show: There exists $D \in \mathcal{X}$ such that

$$D \times_X D \subseteq V.$$

Let $\varepsilon \in \mathbb{R}_{>0}$ such that $V \supseteq B_\varepsilon$.

Let $D = B_{\varepsilon/2}$.

Then $D \supseteq B_{\varepsilon/2}$ so $D \in \mathcal{X}$ and

$$D \times_X D = B_{\varepsilon/2} \times_X B_{\varepsilon/2} \subseteq B_\varepsilon,$$

since $d(x,y) \le d(x,z) + d(z,y) < \varepsilon$ if $(x,y) \in B_{\varepsilon/2} \times_X B_{\varepsilon/2}$.

(3b) Let $(X, \mathcal{X})$ be a uniform space, $E \in \mathcal{X}$ and $x \in X$. By definition, the <u>neighborhood filter of x</u> is

$$\mathcal{N}(x) = \left\{ N \subseteq X \;\middle|\; \begin{array}{l} \text{there exists } V \in \mathcal{X} \\ \text{such that } N \supseteq B_V(x) \end{array} \right\}$$

where

$$B_V(x) = \{ y \in X \mid (x,y) \in V \}.$$

To show: $\mathcal{N}(x) = \{ B_V(x) \mid V \in \mathcal{X} \}$.

To show: (ab) $\mathcal{N}(x) \supseteq \{ B_V(x) \mid V \in \mathcal{X} \}$

(bb) $\mathcal{N}(x) \subseteq \{ B_V(x) \mid V \in \mathcal{X} \}$.

(ab) $\mathcal{N}(x) \supseteq \{ B_V(x) \mid V \in \mathcal{X} \}$ follows directly from the definition of $\mathcal{N}(x)$.

(bb) To show: If $N \in \mathcal{N}(x)$ then there exists $W \in \mathcal{X}$ such that $N = B_W(x)$.

Assume $N \in \mathcal{N}(x)$

Then there exists $V \in \mathcal{X}$ with $N \supseteq B_V(x)$.

To show: There exists $W \in \mathcal{X}$ such that $N = B_W(x)$

Let $W = \{ (y,x) \mid y \in N \}$.

If $(y,x) \in V$ then $y \in B_V(x) \subseteq N$ and so $(y,x) \in W$.

Thus $W \supseteq V$.

Since $V \in \mathcal{X}$ and $W \subseteq X \times X$ and $W \supseteq V$

then $W \in \mathcal{X}$.

To show: $N = B_W(x)$.

To show: (bba) $N \subseteq B_W(x)$

(bbb) $B_W(x) \subseteq N$.

(bba) Assume $n \in N$.

Then $(n,x) \in W$ and $n \in B_W(x)$.

So $N \subseteq B_W(x)$.

(bbb) Assume $y \in B_W(x)$.

Then $(y,x) \in W$.

Thus, by the definition of W, $y \in N$.

So $B_W(x) \subseteq N$.

So $N = B_W(x)$

So $\mathcal{N}(x) = \{B_V(x) \mid V \in \mathcal{X}\}$.

(3c)(ca) To show: $\mathcal{X}_E$ is a uniformity, where

$$\mathcal{X}_E = \{D \subseteq X\times X \mid \text{there exists } k\in\mathbb{Z}_{>0} \text{ with } D \supseteq E_k\}$$

and $E_1, E_2, \ldots \in \mathcal{X}$ are such that

$$\sigma(E_n) = E_n, \quad E_1 \subseteq E \text{ and } E_{n+1} \times_X E_{n+1} \subseteq E_n.$$

To show: (caa) If $D\in\mathcal{X}_E$ then $D\supseteq \Delta(X)$.

(cab) If $D\in\mathcal{X}_E$ and $V\subseteq X\times X$ and $V\supseteq D$

then $V\in\mathcal{X}_E$.

(cac) If $\ell\in\mathbb{Z}_{>0}$ and $V_1, V_2, \ldots, V_\ell\in\mathcal{X}_E$ then

$V_1\cap\cdots\cap V_\ell\in\mathcal{X}_E$

(cad) If $V\in\mathcal{X}_E$ then $\sigma(V)\in\mathcal{X}_E$.

(cae) If $V\in\mathcal{X}_E$ then there exists $D\in\mathcal{X}_E$ such that

$D\times_X D\subseteq V$.

(caa) Assume $D\in\mathcal{X}_E$.

To show: $D\supseteq\Delta(X)$.

There exists $k\in\mathbb{Z}_{>0}$ with $D\supseteq E_k$.

Since $E_k\in\mathcal{X}$ then $E_k\supseteq\Delta(X)$ and $D\supseteq E_k\supseteq\Delta(X)$.

(cab) Assume $D\in\mathcal{X}_E$ and $V\subseteq X\times X$ and $V\supseteq D$.

To show: $V\in\mathcal{X}_E$.

To show: There exists $k\in\mathbb{Z}_{>0}$ with $V\supseteq E_k$.

Let $k \in \mathbb{Z}_{>0}$ such that $D \supseteq E_k$.

Then $V \supseteq D \supseteq E_k$.

(ka c) Assume ~~$k \in \mathcal{X}_E$~~ $\ell \in \mathbb{Z}_{>0}$ and $V_1, V_2, \ldots, V_\ell \in \mathcal{X}_E$

To show: $V_1 \cap V_2 \cap \cdots \cap V_\ell \in \mathcal{X}_E$.

Let $k_1, k_2, \ldots, k_\ell \in \mathbb{Z}_{>0}$ such that $V_j \supseteq E_{k_j}$.

To show: There exists $k \in \mathbb{Z}_{>0}$ such that

$V_1 \cap V_2 \cap \cdots \cap V_\ell \supseteq E_k$

Let $k = \max\{k_1, k_2, \ldots, k_\ell\}$.

If $n \in \mathbb{Z}_{>0}$ then $E_{n+1} \subseteq E_{n+1} \times_X E_{n+1}$ since

if $(x,y) \in E_{n+1}$ then $(x,y) \in E_{n+1}$ and $(y,y) \in E_{n+1}$.

Since $E_{n+1} \times_X E_{n+1} \subseteq E_n$ then $E_{n+1} \subseteq E_n$ and

$E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

Thus $E_k \subseteq E_{k_1}$, $E_k \subseteq E_{k_2}, \ldots, E_k \subseteq E_{k_\ell}$.

So $E_k \subseteq E_{k_1} \cap E_{k_2} \cap \cdots \cap E_{k_\ell}$.

So $V_1 \cap V_2 \cap \cdots \cap V_\ell \supseteq E_{k_1} \cap \cdots \cap E_{k_\ell} \supseteq E_k$.

So $V_1 \cap V_2 \cap \cdots \cap V_\ell \in \mathcal{X}_E$.

(cad) Assume $V \in \mathcal{X}_E$

To show: $\sigma(V) \in \mathcal{X}_E$.

Let $k \in \mathbb{Z}_{>0}$ such that $V \supseteq E_k$.

Since $\sigma(E_k) = E_k$ ~~then~~ then

$$\sigma(V) \supseteq \sigma(E_k) = E_k.$$

So $\sigma(V) \in \mathcal{X}_E$

(cae) Assume $V \in \mathcal{X}_E$.

To show: There exists $D \in \mathcal{X}_E$ such that $D \times_X D \subseteq V$.

Let $k \in \mathbb{Z}_{>0}$ such that $V \supseteq E_k$.

Let $D = E_{k+1}$.

Since $E_{k+1} \times_X E_{k+1} \subseteq E_k$ then

$$D \times_X D = E_{k+1} \times_X E_{k+1} \subseteq E_k \subseteq V.$$

Thus $\mathcal{X}_E$ is a uniformity

To show: $\mathcal{X}_E \subseteq \mathcal{X}$.

To show: If $D \in \mathcal{X}_E$ then $D \in \mathcal{X}$.

Assume $D \in \mathcal{X}_E$.

Let $k \in \mathbb{Z}_{>0}$ such that $D \supseteq E_k$.

Since $E_k \in \mathcal{X}$ and $D \subseteq X \times X$ and $D \supseteq E_k$

then $D \in \mathcal{X}$ (upper ideal condition on $\mathcal{X}$).

(3c) (cb) To show: $\mathcal{X} = \sup\{\mathcal{X}_E \mid E \in \mathcal{X}\}$.

To show: (cba) $\mathcal{X}$ is an upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{X}\}$.

(cbb) If $\mathcal{Y}$ is an upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{X}\}$ then $\mathcal{Y} \supseteq \mathcal{X}$

(cba) To show: If $E \in \mathcal{X}$ then $\mathcal{X}_E \subseteq \mathcal{X}$.

This was established in part (ca) of Question 3.

(cbb) Assume $\mathcal{Y}$ is an upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{X}\}$.

To show: $\mathcal{Y} \supseteq \mathcal{X}$.

We know: If $E \in \mathcal{X}$ then $\mathcal{Y} \supseteq \mathcal{X}_E$.

To show: If $E \in \mathcal{X}$ then $E \in \mathcal{Y}$.

Assume $E \in \mathcal{X}$.

To show: $E \in \mathcal{Y}$.

Since $E_1 = E$ and $E_1 \in \mathcal{X}_E$ then $E \in \mathcal{X}_E$.

Since $\mathcal{X}_E \subseteq \mathcal{Y}$ then $E \in \mathcal{Y}$.

So $\mathcal{X} \subseteq \mathcal{Y}$.

So $\mathcal{X}$ is a least upper bound of $\{\mathcal{X}_E \mid E \in \mathcal{X}\}$.

So $\mathcal{X} = \sup\{\mathcal{X}_E \mid E \in \mathcal{X}\}$.

(3d) Since $\sigma(\mathcal{U}_n) = \mathcal{U}_n$ then $(y,x) \in \mathcal{U}_n$ if $(x,y) \in \mathcal{U}_n$.

So $g_E(y,x) = g_E(x,y)$

Since $g(x,y) \in \{0, 1, 2^{-k} \mid k \in \mathbb{Z}_{>0}\}$ then $g_E(x,y) \in \mathbb{R}_{\geq 0}$.

If $x \in X$ then $(x,x) \in \Delta(X) \subseteq \mathcal{U}_n$ for $n \in \mathbb{Z}_{>0}$.

So $g_E(x,x) = 0$.

For the above, we are, of course, using

$$g_E(x,y) = \begin{cases} 1, & \text{if } (x,y) \notin \mathcal{U}_1, \\ 2^{-k}, & \text{if } (x,y) \in \mathcal{U}_1, (x,y) \in \mathcal{U}_2, \ldots, (x,y) \in \mathcal{U}_k \text{ and } (x,y) \notin \mathcal{U}_{k+1}, \\ 0, & \text{if } (x,y) \in \mathcal{U}_n \text{ for } n \in \mathbb{Z}_{>0}. \end{cases}$$

the definition of $g_E : X \times X \to \mathbb{R}$.

(3f) Let $X = \{0, 1\}$. Let $d : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ be given by

$d(0,0) = 0, \quad d(1,1) = 0, \quad d(1,0) = d(1,0) = 0$

If $\varepsilon \in \mathbb{R}_{>0}$ then $B_\varepsilon(0) = \{0,1\}$ and $B_\varepsilon(1) = \{0,1\}$.

So the pseudo metric space topology on X is

$\mathcal{T} = \{\phi, X\}$

If $\alpha \in \mathbb{R}_{>0}$ and $d_\alpha : X \times X \to \mathbb{R}_{\geq 0}$ is given by

$d_\alpha(0,0) = d_\alpha(1,1)$ and $d_\alpha(0,1) = d_\alpha(1,0) = \alpha$

then $B_{\alpha/2}(0) = \{0\}$ and $B_{\alpha/2}(1) = \{1\}$ so that

$\mathcal{T}_\alpha = \{\phi, \{0\}, \{1\}, X\}$ is the metric space topology on X for the metric d_α.