Assignment 2

MAST30026 Metric and Hilbert Spaces
Semester II 2017
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to be turned in before 2pm on 12 October 2016

(1) (ℓ^p spaces) Let $p \in \mathbb{R}_{\geqslant 1}$.

(a) Carefully define ℓ^p.

(b) Prove that ℓ^p is a normed vector space.

(c) Prove that ℓ^p is not an inner product space unless $p = 2$.

(2) (The p-norms on $V = \mathbb{R}^2$) Let $p \in \mathbb{R}_{>0}$.

(a) Carefully define the p-norm $\| \|_p$ on $\mathbb{R}^2$.

(b) Draw pictures of the open ball of radius 1 centred at $(0,0)$ in the spaces $(\mathbb{R}^2, \| \|_3)$, $(\mathbb{R}^2, \| \|_2)$,$(\mathbb{R}^2, \| \|_1)$,$(\mathbb{R}^2, \| \|_{\frac{1}{2}})$, and $(\mathbb{R}^2, \| \|_{\frac{1}{3}})$,

(c) Show that all the spaces $(\mathbb{R}^2, \| \|_p)$ have the same topology.

(d) Let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}$ be a linear functional, say $\varphi(x_1, x_2) = ax_1 + bx_2$. Give a direct proof that

(i) If $\mathbb{R}^2$ has the norm $\| \|_1$ then the corresponding operator norm is $\|\varphi\|_\infty = \max\{|a|, |b|\}$.

(ii) If $\mathbb{R}^2$ has the norm $\| \|_\infty$ then the corresponding operator norm is $\|\varphi\|_1 = |a| + |b|$.

(iii) If $p \in \mathbb{R}_{>1}$ and $\mathbb{R}^2$ has the norm $\| \|_p$ then the corresponding operator norm is
$\|\varphi\|_q = (|a|^q + |b|^q)^{1/q}$, with $\frac{1}{p} + \frac{1}{q} = 1$.

(3) (The p-adic numbers) Let $p \in \mathbb{Z}_{>0}$ be prime. Define

$$\mathbb{Q}_p = \{a_\ell p^{-\ell} + a_{-\ell+1} p^{-\ell+1} + \cdots \mid \ell \in \mathbb{Z},\ a_i \in \{0, 1, \ldots, p-1\}\}, \qquad \text{and}$$
$$\mathbb{Z}_p = \{a_0 + a_1 p + a_2 p^2 + \cdots \mid a_i \in \{0, 1, \ldots, p-1\}\}.$$

Review the definition of addition, multiplication and $| |_p$ on $\mathbb{Q}_p$.

(a) Show that if $x \in \mathbb{Q}_p$ then there exists $-x \in \mathbb{Q}_p$.

(b) Prove that if $x \in \mathbb{Z}_p$ then $-x \in \mathbb{Z}_p$.

(c) Show that if $x \in \mathbb{Q}_p$ and $x \neq 0$ then there exists $x^{-1} \in \mathbb{Q}_p$.

(d) Show that there exists $z \in \mathbb{Q}_p$ such that $z^{p-1} = 1$.

(e) Prove that $| \ |_p \colon \mathbb{Q}_p \to \mathbb{Q}_p$ is a metric on $\mathbb{Q}_p$.

(f) Show that $\mathbb{Z}_p$ is a compact subset of $\mathbb{Q}_p$.

(g) Show that $\mathbb{Z}_p$ is a open subset of $\mathbb{Q}_p$.

(4) (Power iteration) Let

$$A = \begin{pmatrix} 7465 & 1288 & -1058 & 528 & -222 & 30 \\ -25894 & -4218 & 3676 & -1839 & 744 & -120 \\ -127642 & -20896 & 18119 & -9061 & 3677 & -581 \\ -386350 & -64150 & 54820 & -27400 & 11225 & -1705 \\ -200742 & -32626 & 28500 & -14259 & 5760 & -935 \\ 66848 & 11264 & -9480 & 4736 & -1960 & 285 \end{pmatrix}$$

(a) Use Wolfram alpha or some analogous to determine the eigenvalues and the Jordan form of A. Carefully and thoroughly document the steps that you take and your intermediate results.

(b) For each eigenvalue λ of A compute the matrices

$$\tfrac{1}{\lambda}A - 1, \quad \left(\tfrac{1}{\lambda}A - 1\right)^2, \quad \left(\tfrac{1}{\lambda}A - 1\right)^3 \quad \text{and} \quad E_\lambda = \lim_{n\to\infty} (\tfrac{1}{\lambda}A - 1)^n.$$

(c) Show that $E_\lambda^2 = E_\lambda$, $(1 - E_\lambda)^2 = (1 - E_\lambda)$.

(d) Let V be a separable Hilbert space and let $T\colon V \to V$ be a diagonalisable linear transformation with real eigenvalues $\lambda_1 \geqslant \lambda_2 \geqslant \cdots$. Let λ be the largest eigenvalue of T and let

$$P\colon V \to V \quad \text{be defined by} \quad P = \lim_{n\to\infty} \left(\tfrac{1}{\lambda}T - 1\right)^n.$$

Show that $PV = V_\lambda$, the λ-eigenspace of V.

(5) (Baire category theorem)

(a) Carefully state the Baire theorem for open dense sets.

(b) Carefully prove the Baire theorem for open dense sets.

(c) Give an example illustrating and illuminating the Baire theorem for open dense sets.

(6) (Schauder bases are total sets) A metric space (X, d) is *separable* if it has a countable dense subset. Let $(V, \| \ \|)$ be a normed vector space. A *total set* in a normed vector space $(V, \| \ \|)$ is a subset $A \subseteq V$ such that $\overline{\mathbb{K}\text{-span}(A)} = V$. Let e_i be the vector in $\mathbb{R}^\infty$ with 1 in the ith coordinate and 0 elsewhere.

(a) Show that a Schauder basis of V is a total set of V.

(b) Show that if V has a Schauder basis then V is separable.

(c) Show that if $p \in \mathbb{R}_{\geqslant 1}$ then $(e_1, e_2, \ldots)$ is a Schauder basis of ℓ^p.

(d) Show that $(e_1, e_2, \ldots)$ is not a Schauder basis of ℓ^∞.

(7) (Gram Schmidt) Prove that $(a_1, a_2, \ldots)$ is an orthonormal sequence. Prove that the denominator is the nth principal minor of A. Let V be an inner product space.

(a) Show that an orthonormal sequence $(a_1, a_2, \ldots)$ in V is linearly independent.

(b) Let $(v_1, v_2, \ldots)$ be a sequence of linearly independent vectors in V and let

$$a_1 = \frac{v_1}{\|v_1\|}, \qquad \text{and} \qquad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n+1}, a_n\rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n+1}, a_n\rangle a_n\|}.$$

Show that $(a_1, a_2, \ldots)$ is an orthonormal sequence of linearly independent vectors in V.

(c) Show that

$$\|v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n+1}, a_n\rangle a_n\| = \det(A_n),$$

where $A_n = \big(\langle v_i, v_j\rangle\big)_{1\leqslant i,j\leqslant n}$.

(8) (matrices of linear transformations and adjoints) Let $n \in \mathbb{Z}_{>0}$ and let V be a Hilbert space with orthonormal basis $e_1, e_2, \ldots, e_n$. Let $T\colon V \to V$ be a linear transformation. Let A be the matrix of T with respect to the basis $e_1, \ldots, e_n$ and let B be the matrix of T^* with respect to the basis $e_1, \ldots, e_n$. Let a_{ij} be the (i, j)-entry of A. Show that

(a) Show that $B = \overline{A}^t$.

(b) Show that T is Hermitian if and only if $A = \overline{A}^t$.

(c) Show that T is unitary if and only if $A\overline{A}^t = 1$.

(d) Show that T is self adjoint if and only if $A = \overline{A}^t$.

(e) Show that T is positive if and only if A satisfies

$$\text{If } k \in \{1, \ldots, n\} \text{ and } A_k = \big(a_{ij}\big)_{1\leqslant i,j\leqslant k} \text{ then } \quad \det(A_k) \in \mathbb{R}_{\geqslant 0}.$$

(f) Show that $\|T^*\| = \|T\|$.