Tutorial 11

The eigenvalues λ of A are the roots of the ***characteristic equation***

$$\det(A - \lambda I) = 0$$

The corresponding eigenvectors are the non-zero vectors $\mathbf{v}$ satisfying

$$A\mathbf{v} = \lambda\mathbf{v} \quad \text{or} \quad (A - \lambda I)\mathbf{v} = \mathbf{0}.$$

An $n \times n$ matrix A with n linearly independent eigenvectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ can be ***diagonalized*** according to the formula

$$D = P^{-1}AP$$

where

$$P = [\mathbf{v}_1 \ \mathbf{v}_2 \cdots \mathbf{v}_n] \qquad D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Diagonalization is possible if and only if A has n linearly independent eigenvectors.

Q1. Let

$$A = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(a) By calculating $A\mathbf{v}$, verify that the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A.

(b) Explain why $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is not an eigenvector of A.

Q2. For each of the matrices below answer the following questions:

(a) Find the eigenvalues and corresponding eigenvectors for each matrix.
(b) Is the matrix diagonalizable? Why or why not?
(c) If the matrix, M, is diagonalizable write down a diagonal matrix D and a change of basis matrix P such that

$$D = P^{-1}MP.$$

Check that $PD = MP$.

NOTE: Keep a record of your answers for (i) and (iii) as you will need these later.

(i) $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \end{bmatrix}$ (ii) $B = \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$ (iii) $C = \begin{bmatrix} 4 & -2 & -1 \\ -2 & 4 & -1 \\ -1 & -1 & 1 \end{bmatrix}$

An application of the diagonalization formula is the computation of powers of the matrix A. We have

$$A^k = PD^kP^{-1} \text{ for all integers } k \geq 1.$$

Q3. For the matrix A in Q2(i) compute A^k where $k \geq 1$ is an integer.
What happens to A^k as k approaches infinity?

A matrix A is ***symmetric*** if $A^T = A$. A matrix Q is ***orthogonal*** if $Q^T = Q^{-1}$.

Symmetric matrices with real entries are diagonalizable. In fact, for such matrices we can construct an *orthonormal* set of eigenvectors $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$. The matrix

$$Q = [\mathbf{u}_1 \ \mathbf{u}_2 \cdots \mathbf{u}_n]$$

is then an orthogonal matrix. The diagonalization formula reads

$$D = Q^TAQ$$

where D is the diagonal matrix of the eigenvalues as usual.

Q4. The matrix C in Q2(iii) is symmetric. Modify the eigenvectors obtained in Q2(iii) to find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of C.

Hence find an orthogonal matrix Q such that $Q^TCQ = D$ where D is diagonal. Write down Q^{-1}.

If you finish the above problems before the end of class go on with Exercises 203 to 216 on Topic 7 from your Exercise booklet. You should aim to finish all these problems as soon as possible after your tutorial and to finish all the exercises as soon as possible after your last lecture.

Continued over page - an extra problem for you to work on at home

The standard form of the equation of an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, while the standard form of the equation of a hyperbola is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.
By rotating the axes formed by the standard basis in $\mathbb{R}^2$, to the axes formed by the orthogonal unit vectors $\mathbf{v}_1$, $\mathbf{v}_2$ with coordinates (u, v) so that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},$$

the equation $ax^2 + bxy + cy^2 = d$ can be reduced to standard form

$$\lambda_1 u^2 + \lambda_2 v^2 = d \quad \text{or} \quad \frac{u^2}{(d/\lambda_1)} + \frac{v^2}{(d/\lambda_2)} = 1,$$

and so in the coordinate system (u, v) be identified as an ellipse or hyperbola. Introducing the symmetric matrix

$$A = \begin{bmatrix} a & b/2 \\ b/2 & c \end{bmatrix}$$

the new axes — called the *principal axes* — are in the directions of the normalized eigenvectors of A while λ_1, λ_2 are its eigenvalues.

Q5. (a) Verify that the symmetric matrix

$$\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}$$

has eigenvalues $\lambda = 3, -2$ with corresponding normalized eigenvectors $\frac{1}{\sqrt{5}}(1, 2)$ and $\frac{1}{\sqrt{5}}(-2, 1)$.

(b) Use (a) to identify the conic

$$-x^2 + 4xy + 2y^2 = 3,$$

and specify its principal axes.