3.2. Tutorial sheet, week of 08.08.2016

Review the page: http://www.ms.unimelb.edu.au/~ram/thoughtsandadvice.html , particularly the section on How to study for and take an exam.

Sample exam 1: MAST30026 Metric and Hilbert Spaces Sem II 2016

(1) (the spaces ℓ^p)
  (a) Carefully define the spaces ℓ^p.
  (b) Show that $\ell^1 \subseteq \ell^2 \subseteq \ell^3$.
  (c) Show that $\ell^1 \neq \ell^2 \neq \ell^3$.

(2) (complex bilinear positive definite forms) Let $V = \mathbb{C}\text{-span}\{e_1, e_2\}$ so that $V \cong \mathbb{C}^2$. Show that there exists $f\colon V \times V \to \mathbb{C}$ such that
  (a) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w \in V$ then $f(c_1v_1 + c_2v_2, w) = c_1 f(v_1, w) + c_2 f(v_2, w)$.
  (b) If $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1 + c_2w_2) = c_1 f(v, w_1) + c_2 f(v, w_2)$.
  (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geqslant 0}$,
  (d) If $v \in V$ and $v \neq 0$ then there exists $w \in V$ such that $f(v, w) \neq 0$.

(3) (examples of topologies)
  (a) Carefully define (in English) a topology.
  (b) Carefully define (in proof machine) a topology.
  (c) Let X be a set with 2 elements. Find (with proof) all topologies on X.

(4) (metric space topology and uniformity) Let (X, d) be a metric space.
  (a) Carefully define the metric space topology and the metric space uniformity on X.
  (b) Prove that the metric space topology is a topology on X.
  (c) Prove that the metric space uniformity is a uniformity on X.
  (d) Prove that the uniform space topology of X with the metric space uniformity is the same as the metric space topology on X.

(5) (continuous and uniformly continuous functions)
  (a) Let $f\colon X \to Y$ be a uniformly continuous function. Show that $f\colon X \to Y$ is continuous.
  (b) Let X and Y be metric spaces. Show that $f\colon X \to Y$ is a uniformly continuous function if and only if f satisfies

$$\begin{aligned}&\text{if } \epsilon \in \mathbb{R}_{>0} \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that}\\ &\qquad \text{if } x, y \in X \text{ and } d(x, y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon.\end{aligned}$$

  (c) Let X and Y be metric spaces. Show that $f\colon X \to Y$ is a uniformly continuous function if and only if f satisfies

$$\begin{aligned}&\text{if } \epsilon \in \mathbb{R}_{>0} \text{ and } x \in X \text{ then there exists } \delta \in \mathbb{R}_{>0} \text{ such that}\\ &\qquad \text{if } y \in X \text{ and } d(x, y) < \delta \text{ then } \rho(f(x), f(y)) < \epsilon.\end{aligned}$$

(6) (the set $[1, 2]$ and closures) Consider $\mathbb{R}_{\geqslant 0}$, $\mathbb{Q}_{\geqslant 0}$, and $\mathbb{Z}_{\geqslant 0}$ with the standard topology.
  (a) Show that $[1, 2]$ in $\mathbb{R}_{\geqslant 0}$ is not open.
  (b) Show that $[1, 2]$ in $\mathbb{Q}_{\geqslant 0}$ is not open.
  (c) Show that $[1, 2]$ in $\mathbb{Z}_{\geqslant 0}$ is open.
  (d) Give an example of a $B_\epsilon(x)$ in a metric space X such that

$$\overline{B_\epsilon(x)} \neq \{y \in X \mid d(y, x) \leqslant \epsilon\}.$$

3.3. Exercises and examples

(1) (complex positive definite bilinear forms are skew symmetric) Let V be a $\mathbb{C}$-vector space. Let $f\colon V \times V \to \mathbb{C}$ be a function such that
  (a) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w \in V$ then $f(c_1v_1+c_2v_2, w) = c_1f(v_1, w)+c_2f(v_2, w)$.
  (b) If $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1+c_2w_2) = c_1f(v, w_1)+c_2f(v, w_2)$.
  (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geqslant 0}$.
  Show that
  (A) If $v \in V$ then $f(v, v) = -f(v, v)$,
  (B) If $v \in V$ then $f(v, v) = 0$,
  (C) If $v, w \in V$ then $f(v, w) = -f(w, v)$.

(2) (nondegenerate complex positive definite bilinear forms) Let $V = \mathbb{C}\text{-span}\{e_1, e_2\}$ so that $V \cong \mathbb{C}^2$. Show that there exists $f\colon V \times V \to \mathbb{C}$ such that
  (a) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w \in V$ then $f(c_1v_1+c_2v_2, w) = c_1f(v_1, w)+c_2f(v_2, w)$.
  (b) If $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1+c_2w_2) = c_1f(v, w_1)+c_2f(v, w_2)$.
  (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geqslant 0}$,
  (d) If $v \in V$ and $v \neq 0$ then there exists $w \in V$ such that $f(v, w) \neq 0$.

(3) (complex symmetric positive definite bilinear forms are zero) Let V be a $\mathbb{C}$-vector space. Let $f\colon V \times V \to \mathbb{C}$ be a function such that
  (a) If $c_1, c_2 \in \mathbb{C}$ and $v_1, v_2, w \in V$ then $f(c_1v_1+c_2v_2, w) = c_1f(v_1, w)+c_2f(v_2, w)$.
  (b) If $c_1, c_2 \in \mathbb{C}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1+c_2w_2) = c_1f(v, w_1)+c_2f(v, w_2)$.
  (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geqslant 0}$.
  (d) If $v, w \in V$ then $f(v, w) = f(w, v)$.
  Show that if $v, w \in V$ then $f(v, w) = 0$.

(4) (nondegenerate real positive definite symmetric bilinear forms) Give an example of a nonzero $\mathbb{R}$-vector space and a function $f\colon V \times V \to \mathbb{R}$ such that
  (a) If $c_1, c_2 \in \mathbb{R}$ and $v_1, v_2, w \in V$ then $f(c_1v_1+c_2v_2, w) = c_1f(v_1, w)+c_2f(v_2, w)$.
  (b) If $c_1, c_2 \in \mathbb{R}$ and $w_1, w_2, v \in V$ then $f(v, c_1w_1+c_2w_2) = c_1f(v, w_1)+c_2f(v, w_2)$.
  (c) If $v \in V$ then $f(v, v) \in \mathbb{R}_{\geqslant 0}$.
  (d) If $v, w \in V$ then $f(v, w) = f(w, v)$.
  (e) If $v \in V$ and $v \neq 0$ then there exists $w \in V$ such that $f(v, w) \neq 0$.