Lecture 28: Metric and Hilbert spaces
Eigenvectors and Eigenvalues

Let $(V, \|\ \|_V)$ and $(W, \|\ \|_W)$ be normed vector spaces.

$$B(V,W) = \left\{ T: V \to W \ \middle|\ \begin{matrix} T \text{ is linear and} \\ \|T\| < \infty \end{matrix} \right\}$$

where

$$\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \ \middle|\ v \in V,\ v \neq 0 \right\}$$

$$= \sup\{ \|Tx\| \mid x \in V,\ \|x\| = 1 \}.$$

<u>The adjoint linear operator</u>

Let H_1 and H_2 be Hilbert spaces and $T: H_1 \to H_2$ a bounded linear operator.

The <u>adjoint of T</u> is $T^*: H_2 \to H_1$ given by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for } \cancel{x,y \in} \ \begin{matrix} x \in H_1 \\ y \in H_2. \end{matrix}$$

<u>HW</u>: Show that $T^*: H_2 \to H_1$ is a well defined linear operator.

<u>HW</u>: Show that $\|T^*\| = \|T\|$.

<u>HW</u>: Show that $T^{**} = T$.

Favorite Hilbert spaces

(1) $\mathbb{C}^n$ with

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n.$$

(2) $\mathbb{R}^n$ with

$$\langle (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n.$$

(3) $\ell^2 = \{ (x_1, x_2, \ldots) \in \mathbb{R}^\infty \mid \|x\| < \infty \}$ with

$$\langle (x_1, x_2, \ldots), (y_1, y_2, \ldots) \rangle = \sum_{i \in \mathbb{Z}_{>0}} x_i \bar{y}_i$$

(4) $\ell^2(\mathbb{C}) = \{ (x_1, x_2, \ldots) \in \mathbb{C}^\infty \mid \|x\| < \infty \}$ with

$$\langle (x_1, x_2, \ldots), (y_1, y_2, \ldots) \rangle = \sum_{i \in \mathbb{Z}_{>0}} x_i \bar{y}_i.$$

<u>Diagonal operators</u>: $T: \ell^2 \to \ell^2$ with

$$T(a_1, a_2, \ldots) = (\lambda_1 a_1, \lambda_2 a_2, \ldots)$$

In the "basis" $(e_1, e_2, \ldots)$ the matrix of T is

$$A = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{pmatrix}$$

<u>Shift operators</u>: $T: \ell^2 \to \ell^2$ given by

$$T(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots)$$

In the "basis" $(e_1, e_2, \ldots)$ the matrix of T is

$$A = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \end{pmatrix} \quad \text{and} \quad A^t = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & \ddots \end{pmatrix}$$

is the matrix of T^*. Then

T^* is an isometry and $T^*T = 1$

but $TT^* \neq 1$.

<u>Definitions</u> Let $T: H \to H$ be a linear operator.

T is <u>compact</u> if T satisfies:

if $(x_1, x_2, \ldots)$ is a sequence in $\{x \in H \mid \|x\| = 1\}$ then $(Tx_1, Tx_2, \ldots)$ has a cluster point in H.

T is <u>self adjoint</u> if T satisfies:

if $x, y \in H$ then $\langle Tx, y \rangle = \langle x, Ty \rangle$.

T is an <u>isometry</u> if T satisfies:

if $x, y \in H$ then $\langle Tx, Ty \rangle = \langle x, y \rangle$.

T is <u>unitary</u> if T satisfies:

$$TT^* = T^*T = 1.$$

T is <u>positive</u> if T satisfies:

if $x \in H$ then $\langle Tx, x \rangle \in \mathbb{R}_{\geq 0}$.

<u>Note:</u> If $T: H \to H$ is self adjoint then

$$\langle Tx, x \rangle = \langle x, Tx \rangle = \overline{\langle Tx, x \rangle}$$

so that $\langle Tx, x \rangle \in \mathbb{R}$.

<u>Eigenvalues and Eigenvectors</u>

Let $T: H \to H$ be a linear operator.

The <u>λ-eigenspace of T</u> is

$$X_\lambda = \{ v \in H \mid Tv = \lambda v \} = \ker(T - \lambda)$$

The <u>point spectrum of T</u> is

$$\sigma_p(T) = \{ \lambda \in \mathbb{C} \mid X_\lambda \neq 0 \}.$$

<u>Questions</u>: Let $W = \bigoplus_{\lambda \in \sigma_p(T)} X_\lambda$

(1) When is $W = 0$? When is W small?

(2) When is W dense in H? When is W big?

Ass 2 Q1: If $T: H \to H$ is self adjoint and $X_\lambda \neq 0$ then $\lambda \in \mathbb{R}$.

Ass 2 Q2: If $T: H \to H$ is self adjoint and $\lambda \neq \gamma$ then $X_\lambda \perp X_\gamma$

Ass 2 Q3: If $T: H \to H$ is compact and $\lambda \neq 0$ then $\dim(X_\lambda)$ is finite.

Main Theorem: If H has a countable basis and $T: H \to H$ is bounded self adjoint compact linear operator then

T is a diagonal operator

i.e. there exists an orthonormal basis of H consisting of eigenvectors of T.