Lecture 18: Metric and Hilbert spaces

Let $(X, \mathcal{T})$ be a topological space. Let $A \subseteq X$.

The boundary of A is $\partial A = \overline{A} \cup \overline{A^c}$

The set A is dense in X if $\overline{A} = X$.

The set A is nowhere dense in X if $(\overline{A})^\circ = \phi$

Examples

(a) The boundary of $(0,1]$ in $\mathbb{R}$ is $\{0,1\}$

$\overline{(0,1]} = [0,1]$ ($(0,1]$ is dense in $[0,1]$)

$(0,1]^c = (-\infty, 0] \cup (1,\infty)$

$\overline{(0,1]^c} = (-\infty, 0] \cup [1,\infty)$ and $\partial((0,1]) = \{0,1\}$.

(b) If $E = \mathbb{Q}$ in $\mathbb{R}$ then $\overline{\mathbb{Q}} = \mathbb{R}$, $\overline{\mathbb{Q}^c} = \mathbb{R}$ and

$\partial \mathbb{Q} = \overline{\mathbb{Q}} \cap \overline{\mathbb{Q}^c} = \mathbb{R}$. Since $\overline{\mathbb{Q}} = \mathbb{R}$, $\mathbb{Q}$ is dense in $\mathbb{R}$.

Also $\mathbb{Q}^\circ = \phi$, since $\mathbb{Q}$ has no interior points.

(c) In $\mathbb{R}$, $\overline{\mathbb{Z}} = \mathbb{Z}$ and $\mathbb{Z}^\circ = \phi$, so $(\overline{\mathbb{Z}})^\circ = \phi$.

So $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

(d) In $\mathbb{R}^2$;

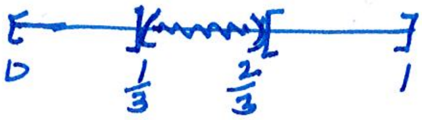
$\overline{\mathbb{R}} = \mathbb{R}$ and $\mathbb{R}^\circ = \phi$, since $\mathbb{R}$ has no interior pts

So $\mathbb{R}$ is nowhere dense in $\mathbb{R}^2$.

The Cantor set

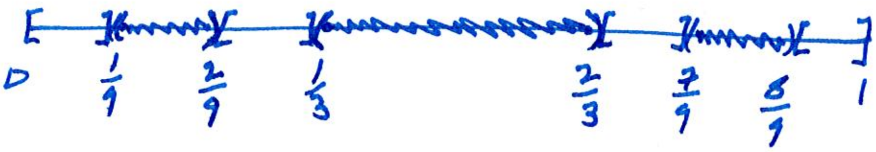
Let $A = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ and remove the middle third of A to get

$A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$



Now remove the middle third of each of the 2 components of A_1 to get

$$A_2 = [1, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1]$$



The Cantor set is obtained by continuing this process:

$$C = \left((\tfrac{1}{3}, \tfrac{2}{3}) \cup (\tfrac{1}{9}, \tfrac{2}{9}) \cup (\tfrac{7}{9}, \tfrac{8}{9}) \cup (\tfrac{1}{27}, \tfrac{2}{27}) \cup (\tfrac{7}{27}, \tfrac{8}{27}) \cup \ldots \right)^c$$

where the complement is taken in $[0,1]$.

HW: Show that C is nowhere dense

HW: Show that $\mathrm{Card}(C) = \mathrm{Card}(\mathbb{R})$.

<u>Theorem</u> Let (X, d) be a complete metric space.
Let $U_1, U_2, \ldots$ be open dense subsets of X.
Then

$$\left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n\right) = U_1 \cap U_2 \cap \cdots \quad \text{is dense in } X.$$

<u>Proof</u>: To show: $\overline{\left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n\right)} = X$.

To show: If $x \in X$ then x is a close point to $\bigcap_{n \in \mathbb{Z}_{>0}} U_n$.

Assume $x \in X$.
To show: x is a close point to $\bigcap_{n \in \mathbb{Z}_{>0}} U_n$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $B_\varepsilon(x) \cap \left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n\right) \neq \emptyset$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.
To show: There exists $y \in B_\varepsilon(x) \cap \left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n\right)$.

Let $x_0 = x$ and $\varepsilon_0 = \varepsilon$. Using that U_n is open and dense let $x_k \in X$ and $\varepsilon_{k+1} \in \mathbb{R}_{>0}$ be such that

$$B_{3\varepsilon_{k+1}}(x_{k+1}) \subseteq B_{\varepsilon_k}(x_k) \cap U_{k+1} \quad \text{and} \quad \varepsilon_{k+1} < \frac{\varepsilon_k}{3}.$$

Let $y = \lim_{k \to \infty} x_k$.

To show: (a) y exists.

(b) $y \in B_\varepsilon(x) \cap (U_1 \cap U_2 \cap \cdots)$

and the rest of the proof is nicely exposited in §10.23 of the Notes. //.

<u>Baire Theorem – Nowhere dense version</u>

Let (X, d) be a complete metric space.
Let $F_1, F_2, \ldots$ be nowhere dense subsets of X.
Show that

$$(F_1 \cup F_2 \cup \cdots)^\circ = \phi.$$

<u>Uniform boundedness theorem</u>

Let (X, d) be a complete metric space.
Let $f_1, f_2, \ldots$ be sequence of continuous functions

$$f_n : X \to \mathbb{R}.$$

Assume that

if $x \in X$ then $\{f_1(x), f_2(x), \ldots\}$ is bounded.

Show that there exists and open set U in X and $M \in \mathbb{R}_{>0}$ such that

if $x \in U$ and $n \in \mathbb{Z}_{>0}$ then $|f_n(x)| \leq M$.