Metric and Hilbert Spaces Assignment 2 Solutions Question 1(a).

(1a) Let V and W be $\mathbb{K}$-vector spaces. 2014

A <u>linear operator</u>, or <u>linear transformation</u>, from V to W is a function $T: V \to W$ such that

(a) If $v_1, v_2 \in V$ then $T(v_1+v_2) = T(v_1) + T(v_2)$

(b) If $v \in V$ and $c \in \mathbb{K}$ then $T(cv) = cT(v)$.

Let V be a $\mathbb{K}$-vector space and let $T: V \to V$ be a linear operator. Let $\mu \in \mathbb{K}$.

An <u>eigenvector</u> of T with <u>eigenvalue</u> μ is a vector $v \in V$ such that $Tv = \mu v$.

(1b) There does not always exist $v \in V$ with $v \neq 0$ which is an eigenvector of $T: V \to V$.

An example is

$$V = \mathbb{C}\text{-span}\{e_i \mid i \in \mathbb{Z}\} = \mathbb{C}\text{span}\{\ldots, e_{-1}, e_0, e_1, \ldots\}$$

with $T(e_i) = e_{i-1}$.

If $v = \sum_{i \in \mathbb{Z}} c_i e_i$, with all but a finite number of the c_i equal to 0, and $Tv = \mu v$

then $\sum_{j \in \mathbb{Z}} c_j e_{j-1} = \sum_{i \in \mathbb{Z}} \mu c_i e_i$, giving $\mu c_i = c_{i+1}$.

If $\mu \neq 0$ then all c_i must be 0 and $v = 0$.

If $\mu = 0$ then $Tv = \sum_{j\in\mathbb{Z}} c_j e_{j-1} = 0_V$ and $v = 0$.

So $v = 0$.

However, if <u>V is finite dimensional</u> then $T: V \to V$ <u>does</u> have a nonzero eigenvector.

A vector v is an eigenvector of eigenvalue μ exactly when $Tv = \mu v$, so that

$$0 = (T - \mu I)v \quad \text{and} \quad v \in (\text{null space of } T - \mu I)$$

If $\det(T - \mu I) = 0$ then the null space is nonzero and T has an eigenvector of eigenvalue μ.

Then $\det(T - \mu I) = 0$ if μ is a root of $\det(T - tI)$, the characteristic polynomial of T.

Since $\mathbb{C}$ is algebraically closed,

$\det(T - tI)$ has a root μ in $\mathbb{C}$.

So the null space of $T - \mu I$ is nonzero.

So T has an eigenvector of eigenvalue μ.

(c) We need to determine when $\ker(A-\lambda) \neq 0$

We need
$$(A-\lambda)\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$
to have nonzero solutions.

We solve this by row reducing $A-\lambda$, where

$$A-\lambda = \begin{pmatrix} 1 & 5 & -2 \\ 6 & 0 & 2 \\ \pi & \sqrt{7} & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 5 & -2 \\ 6 & -\lambda & 2 \\ \pi & \sqrt{7} & -\lambda \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 5 & -2 \\ 6 & -\lambda & 2 \\ \pi & \sqrt{7} & -\lambda \end{pmatrix} \xrightarrow[x_{23}\left(\frac{-6}{\pi}\right)]{x_{13}\left(\frac{-(1-\lambda)}{\pi}\right)} \begin{pmatrix} 0 & 5-\frac{(1-\lambda)}{\pi}\sqrt{7} & -2-\frac{(1-\lambda)}{\pi}(-\lambda) \\ 0 & -\lambda-\frac{6}{\pi}\sqrt{7} & 2+\frac{6}{\pi}\lambda \\ \pi & \sqrt{7} & -\lambda \end{pmatrix}$$

If $\ker(A-\lambda) \neq 0$ then the next step of the row reduction will produce a row of zeros,

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda-\frac{6}{\pi}\sqrt{7} & 2+\frac{6}{\pi}\lambda \\ \pi & \sqrt{7} & -\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

A solution of this system is

$$v_3 = 1, \quad v_2 = \frac{-(2+\frac{6}{\pi}\lambda)}{-\lambda-\frac{6}{\pi}\sqrt{7}}, \quad v_1 = \frac{1}{\pi}\left(+\lambda-\sqrt{7}\,v_2\right) \qquad (*)$$
$$= \frac{\lambda}{\pi} - \frac{\sqrt{7}\,(2+\frac{6}{\pi}\lambda)}{\pi(\lambda+\frac{6}{\pi}\sqrt{7})}.$$

This vector $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ will be an eigenvector of

$\begin{pmatrix} 1 & 5 & -2 \\ 6 & 0 & 2 \\ \pi & \sqrt{7} & 0 \end{pmatrix}$ whenever $\det(\lambda - A) = 0$.

Let us approximate λ such that $\det(\lambda - A) = 0$.

$$\det(\lambda - A) = \det \begin{pmatrix} \lambda-1 & -5 & +2 \\ -6 & \lambda & -2 \\ -\pi & -\sqrt{7} & \lambda \end{pmatrix}$$

$$= (\lambda-1)(\lambda^2 - 2\sqrt{7}) + 5(-6\lambda - 2\pi) + (+2)(6\sqrt{7} + \pi\lambda)$$

$$= \lambda^3 - \lambda^2 - 2\sqrt{7}\lambda + 2\sqrt{7} - 30\lambda - 10\pi + 2\pi\lambda + 12\sqrt{7}$$

$$= \lambda^3 - \lambda^2 + (2\pi - 2\sqrt{7} - 30)\lambda + (14\sqrt{7} - 10\pi)$$

$$= \lambda^3 - \lambda^2 + a\lambda + b, \text{ where } a = 2\pi - 2\sqrt{7} - 30$$
$$b = 14\sqrt{7} - 10\pi.$$

Now $\sqrt{7} \approx 2.65$ since $26^2 = 676$ and $27^2 = 729$

So $a \approx 2(3.14 - 2.65) - 30 = 2(.49) - 30 = .98 - 30 = -29.02$

$b \approx 14 \cdot 2.65 - 10 \cdot 3.14 = 2(18.55 - 15.70) = 2(2.85) = 5.70$

So our equation is approximately

$$\lambda^3 - \lambda^2 - 29.02\lambda + 5.7 = 0.$$

When $\lambda = 1$ then $1^3 - 1^2 - 30 \cdot 1 + 6 = -24$

$\lambda = 2$ then $2^3 - 2^2 - 30 \cdot 2 + 6 = -50$

$\lambda = 3$ then $27 - 9 - 90 + 6 = 24 - 90 = -66$

$\lambda = 4$ then $16 \cdot 3 - 120 + 6 = 54 - 120 = -66$

$\lambda = 5$ then $25 \cdot 4 - 150 + 6 = 106 - 150 = -44$

$\lambda = 6$ then $36 \cdot 5 - 180 + 6 = 186 - 180 = 6$

So it looks like there is a ~~zero~~ around $\lambda = 6$.

Indeed, Wolfram alpha

`solve x^3-x^2+(2*pi-2*sqrt(7)-30)x+(14*sqrt(7)-10*pi)=0`

reports

$$x \approx -5.01186, \quad x \approx 0.192861, \quad x \approx 5.819.$$

and Wolfram alpha

`eigenvalues {{1,5,-2},{6,0,2},{pi,sqrt(7),0}}`

reports

$$\lambda_1 \approx 5.819, \quad \lambda_2 \approx -5.01186, \quad \lambda_3 \approx 0.192861$$

With $\lambda = 6$, the formulas for v_1, v_2, v_3 given in (*)

produce

$$v_1 = \frac{6}{\pi} - \frac{\sqrt{7}}{\pi}\frac{\left(2+\frac{6}{\pi}\cdot 6\right)}{\left(6+\frac{6}{\pi}\cdot\sqrt{7}\right)} \approx \frac{6}{3.14} - \frac{2.65}{3.14}\frac{\left(2+\frac{36}{3.14}\right)}{\left(6+\frac{6}{3.14}\cdot 2.65\right)}$$

$$\approx 1.9 - 0.7\frac{(2+12)}{(6+5)} \approx 1.9 - 0.7\cdot\frac{14}{11} \approx 2-1 = 1.$$

$$v_2 = -\frac{\left(2+\frac{6}{\pi}\cdot 6\right)}{-6-\frac{6}{\pi}\sqrt{7}} \approx \frac{2+12}{6+5} = \frac{14}{11} \approx 1.28$$

$$v_3 = 1$$

and, Wolfram alpha reports

$$(v_1, v_2, v_3) \approx (0.836448, 1.20617, 1)$$

which indicates that our rough approximation was in the right ballpark.

(1d) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation corresponding to the matrix

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ with respect to the basis

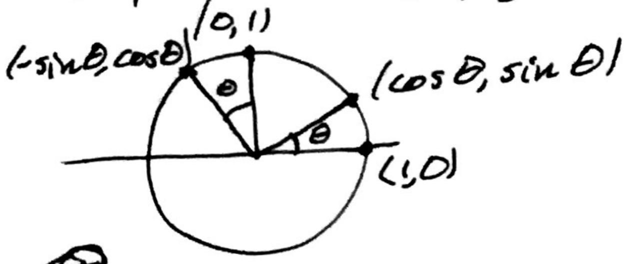
$\{e_1, e_2\}$ where $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then $\det(A - tI) = \det\begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2 + 1$,

which has roots $\pm\sqrt{-1} \notin \mathbb{R}$. So A does not have an eigenvector in $\mathbb{R}^2$.

More generally, if $\theta \in \{x \in \mathbb{R} \mid 0 \le x < 2\pi\}$ then

$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}$



is a rotation by an angle θ.

If θ is not an integer multiple of π then there is <u>no</u> <u>nonzero</u> <u>vector</u> in $\mathbb{R}^2$ such that rotation of v by θ is a multiple of v.

So if θ is not an integer multiple of π then $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix}$ has no nonzero eigenvector in $\mathbb{R}^2$.