2016

Metric and Hilbert Spaces: Assignment 1 Solutions (7a)

(7a) Following §3.5.2(1) and §3.5.2(2) we will use the following definitions of continuous and uniformly continuous functions (between metric spaces) for this question.

Let (X,d) and (Y,ρ) be metric spaces and let $f: X \to Y$ be a function.

The function $f: X \to Y$ is continuous if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ and $x \in X$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.

The function $f: X \to Y$ is uniformly continuous if f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ ~~and~~ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in X$ and $d(x,y) < \delta$ then $\rho(f(x), f(y)) < \varepsilon$.

(7b) Let $n \in \mathbb{Z}_{>0}$.

To show: $x^n : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{\ge 0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in \mathbb{R}_{\ge 0}$ and $d(x,y) < \delta$ then $d(x^n, y^n) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{\ge 0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that

if $y \in \mathbb{R}_{\ge 0}$ and $d(x,y) < \delta$ then $d(x^n, y^n) < \varepsilon$.

Let $\delta = \min\left\{1, \frac{\varepsilon}{2} \cdot \frac{1}{(x+1)^{n-1} \cdot n}\right\}$

To show: If $y \in \mathbb{R}_{\ge 0}$ and $d(x,y) < \delta$ then $d(x^n, y^n) < \varepsilon$.

Assume $y \in \mathbb{R}_{\ge 0}$ and $d(x,y) < \delta$.

To show: $d(x^n, y^n) < \varepsilon$.

To show: $|y^n - x^n| < \varepsilon$.

$$|y^n - x^n| = |(y-x)(y^{n-1} + y^{n-2}x + y^{n-3}x^2 + \cdots + yx^{n-2} + x^{n-1})|$$

Since $y < x + \delta$ then

$$|y^n - x^n| \le |\delta((x+\delta)^{n-1} + (x+\delta)^{n-2}x + \cdots + x^{n-1})|$$

$$\le \delta \cdot |(x+\delta)^{n-1} + (x+\delta)^{n-1} + \cdots + (x+\delta)^{n-1}|$$

$$= \delta \cdot (x+\delta)^{n-1} \cdot n \le \delta (x+1)^{n-1} \cdot n < \varepsilon$$

$$\cancel{\le \delta x^{n-1} n < \varepsilon}$$

So $x^n : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$ is continuous.

MATH Ass1. Solns 23.09.2016 (7c)

(7c) Let $n \in \mathbb{Z}_{>1}$.

To show: $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is not uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ ~~then there exists~~ and $\delta \in \mathbb{R}_{>0}$ then there exist $x, y \in \mathbb{R}_{\geq 0}$ with $d(x,y) < \delta$ and $d(x^n, y^n) > \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$ and $\delta \in \mathbb{R}_{>0}$.

To show: There exist $x, y \in \mathbb{R}_{\geq 0}$ such that $d(x,y) < \delta$ and $d(x^n, y^n) > \varepsilon$.

Let $x = \left(\frac{2}{n\delta}(\varepsilon+1)\right)^{\frac{1}{n-1}}$ and let $y = x + \frac{\delta}{2}$.

To show: (ca) $d(x,y) < \delta$

(cb) $d(x^n, y^n) > \varepsilon$.

(ca) Since $y - x = x + \frac{\delta}{2} - x = \frac{\delta}{2}$ then $d(x,y) < \delta$.

(cb) To show: $d(x^n, y^n) > \varepsilon$.

$$d(x^n, y^n) = |y^n - x^n| = \left| \left(x + \tfrac{\delta}{2}\right)^n - x^n \right|$$

$$= \left| \left(x + \tfrac{\delta}{2} - x\right)\left(\left(x+\tfrac{\delta}{2}\right)^{n-1} + \left(x+\tfrac{\delta}{2}\right)^{n-2} x + \cdots + \left(x+\tfrac{\delta}{2}\right)x^{n-2} + x^{n-1} \right) \right|$$

$$\geq \frac{\delta}{2} \cdot (x^{n-1} + x^{n-1} + \cdots + x^{n-1}) = \frac{\delta}{2} \cdot n x^{n-1}$$

$$= \frac{\delta}{2} n \left(\left(\frac{2}{n\delta}(\varepsilon+1) \right)^{\frac{1}{n-1}} \right)^{n-1} = \frac{\delta}{2} n \cdot \frac{2}{n\delta}(\varepsilon+1) = \varepsilon + 1 > \varepsilon.$$

So $x^n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is not uniformly continuous.

M&H Ass 1
Solns 23.09.2016 (7d)

(7d) To show: (da) $x^0: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is uniformly continuous.

(db) $x^1: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is uniformly continuous.

(da) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(x^0, y^0) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(x^0, y^0) < \varepsilon$.

Let $\delta = 1$.

To show: If $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(x^0, y^0) < \varepsilon$.

Assume $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$.

To show: $d(x^0, y^0) < \varepsilon$.

$$d(x^0, y^0) = |y^0 - x^0| = |1 - 1| = 0 < \varepsilon,$$

since $\varepsilon \in \mathbb{R}_{>0}$.

So $x^0: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is uniformly continuous.

(db) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(x^1, y^1) < \varepsilon$.

M&H Ass 1
Solns 23.09.2016 (7d)

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that
if $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(x', y') < \varepsilon$.

Let $\delta = \varepsilon$

To show: If $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(x', y') < \varepsilon$.

Assume $x, y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$.

To show: $d(x', y') < \varepsilon$.

$$d(x', y') = d(x, y) < \delta = \varepsilon.$$

So $d(x', y') < \varepsilon$ and $x' : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is uniformly continuous.

(7e) To show: $e^x : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{\geq 0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(e^x, e^y) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}_{\geq 0}$.

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that
if $y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$ then $d(e^x, e^y) < \varepsilon$.

Let $\delta = \dfrac{\varepsilon}{2e^x}$

To show: If $y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$
then $d(e^x, e^y) < \varepsilon$.

M4HAss 1
Solus 23.09.2016 (7c)

Assume $y \in \mathbb{R}_{\geq 0}$ and $d(x,y) < \delta$.

To show: $d(e^x, e^y) < \varepsilon$.

$$d(e^x, e^y) = |e^y - e^x| \cancel{= |e^x}$$

$$= \left| \left(1 + y + \tfrac{1}{2}y^2 + \tfrac{1}{3!}y^3 + \dots\right) - \left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \dots\right) \right|$$

$$= \left| (y-x) + \tfrac{1}{2}(y^2 - x^2) + \tfrac{1}{3!}(y^3 - x^3) + \dots \right|$$

$$= \left| (y-x)\left(1 + \tfrac{1}{2}(y+x) + \tfrac{1}{3!}(y^2 + xy + x^2) + \dots\right) \right|$$

Assume $x < y$, otherwise switch x and y.
Then

$$d(e^x, e^y) \leq \left| (y-x)\left(1 + \tfrac{1}{2}2x + \tfrac{1}{3!}3x^2 + \dots\right) \right|$$

$$\leq \left| \delta \cdot \left(1 + \tfrac{2x}{2} + \tfrac{3}{3!}x^2 + \tfrac{4}{4!}x^3 + \dots\right) \right|$$

$$\leq \delta \cdot \left(1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{3!}x^3 + \dots\right)$$

$$= \delta e^x < \frac{\varepsilon}{2e^x} e^x = \frac{\varepsilon}{2} < \varepsilon.$$