2016
Metric and Hilbert Assignment 1 Solutions

(2a) Assume $a,b,c \in \mathbb{R}_{\geq 0}$ and $a \leq b$.

To show: $a+c \leq b+c$.

By definition of $\leq$ there exists $x \in \mathbb{R}_{\geq 0}$ such that $a+x=b$.

Using associativity and commutativity of addition on $\mathbb{R}_{\geq 0}$,

$$(a+c)+x = a+(c+x) = a+(x+c) = (a+x)+c = b+c.$$

So $a+c \leq b+c$.

(2b) Assume $x,y \in \mathbb{R}_{>0}$.

To show: There exists $n \in \mathbb{Z}_{>0}$ such that $y < nx$.

$$x = x_\ell \left(\frac{1}{10}\right)^\ell + x_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots = x_\ell x_{\ell+1} \cdots x_{-1} x_0 . x_1 x_2 \cdots$$

$$y = y_m \left(\frac{1}{10}\right)^m + y_{m+1}\left(\frac{1}{10}\right)^{m+1} + \cdots = y_m y_{m+1} \cdots y_{-1} y_0 . y_1 y_2 \cdots$$

with $x_\ell \neq 0$ and $y_m \neq 0$ so that $x_\ell, y_m \in \{1,2,\ldots,9\}$.

Then

$$x \geq \left(\frac{1}{10}\right)^\ell > \left(\frac{1}{10}\right)^{\ell+1}$$

and

$$y \leq \left(\frac{1}{10}\right)^{m-1} < \left(\frac{1}{10}\right)^{m-2}$$

So

$$y < \left(\frac{1}{10}\right)^{m-2} = \left(\frac{1}{10}\right)^{m-2-(\ell+1)} \left(\frac{1}{10}\right)^{\ell+1} < \left(\frac{1}{10}\right)^{m-2-(\ell+1)} x,$$

where we are using that if $a, b, c \in \mathbb{R}_{\geq 0}$
and $a < b$ then there exist $x \in \mathbb{R}_{>0}$ with $a + x = b$
and $ac \cancel{+xc} + xc = (a+x)c = bc$
so that $xc \in \mathbb{R}_{\geq 0}$ and $ac \leq bc$ with
$ac < bc$ if $c \neq 0$.

Let $n = \max\left\{ \left(\frac{1}{10}\right)^{m-l-3}, 1 \right\}$ so that

$n = 10^{l-m+3}$ if $l-m+3 > 0$ and
$n = 1$ if $l-m+3 < 0$.

Then
$$y < \left(\frac{1}{10}\right)^{m-l-3} x \leq nx. \quad \text{So } y < nx.$$

(2c) Assume $a, b \in \mathbb{R}_{\geq 0}$ and $a < b$.
To show: There exists $c \in \mathbb{Q}_{\geq 0}$ such that
$a < c < b$.

Since $a < b$ there exists $x \in \mathbb{R}_{>0}$ such that
$a + x = b$.

Let
$$a = a_l \left(\frac{1}{10}\right)^l + a_{l+1}\left(\frac{1}{10}\right)^{l+1} + \cdots$$
$$x = x_m \left(\frac{1}{10}\right)^m + x_{m+1}\left(\frac{1}{10}\right)^{m+1} + \cdots$$

with $a_l \neq 0$ and $x_m \neq 0$ so that $a_l, x_m \in \{1, \ldots, 9\}$.

Let

$$a_{\le k} = \begin{cases} a_\ell \left(\frac{1}{10}\right)^\ell + a_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots + a_k\left(\frac{1}{10}\right)^k, & \text{if } k \ge \ell, \\ 0, & \text{if } k < \ell \end{cases}$$

Then $a_{\le k} \in \mathbb{Q}_{\ge 0}$.

Let $c = a_{\le (m+2)} + \left(\frac{1}{10}\right)^{m+1}$. Then $c \in \mathbb{Q}_{\ge 0}$.

To show: (ca) $b > c$

(cb) $c > a$.

(ca) To show: $b > c$

$$\begin{aligned} b = a + x &= a + x_m\left(\frac{1}{10}\right)^m + \cdots \\ &\ge a + \left(\frac{1}{10}\right)^m, \quad \text{since } x_m \ge 1, \\ &> a + \left(\frac{1}{10}\right)^{m+1}, \quad \text{since } \left(\frac{1}{10}\right)^{m+1} + 9\left(\frac{1}{10}\right)^{m+1} = \left(\frac{1}{10}\right)^m \\ &\ge a_{\le(m+2)} + \left(\frac{1}{10}\right)^{m+1}, \quad \text{since } a_{\le(m+2)} \le a. \\ &= c \end{aligned}$$

So $b > c$.

(cb) To show: $c > a$

$$c = a_{\leq(m+2)} + \left(\frac{1}{10}\right)^{m+1} > a_{\leq(m+2)} + \left(\frac{1}{10}\right)^{m+2}$$

$$= a_l\left(\frac{1}{10}\right)^l + a_{l+1}\left(\frac{1}{10}\right)^{l+1} + \dots + a_{m+2}\left(\frac{1}{10}\right)^{m+2} + \left(\frac{1}{10}\right)^{m+2}$$

$$\geq a_l\left(\frac{1}{10}\right)^l + \dots + a_{m+2}\left(\frac{1}{10}\right)^{m+2} + a_{m+3}\left(\frac{1}{10}\right)^{m+3} + \dots$$

$$= a.$$

So $c > a$.

So $b > c > a$, with $c = a_{\leq(m+2)} + \left(\frac{1}{10}\right)^{m+1} \in \mathbb{Q}_{\geq 0}$

(2d) Assume $a, b \in \mathbb{R}_{\geq 0}$ and $a < b$.

To show: There exists $c \notin \mathbb{Q}_{\geq 0}$ such that
$$a < c < b.$$

We will use that $\sqrt{2} \in \mathbb{R}_{\geq 0}$ and $\sqrt{2} \notin \mathbb{Q}_{\geq 0}$

and $1 < \sqrt{2} < 10$ (since $1^2 < (\sqrt{2})^2 < 100$ in $\mathbb{Z}_{\geq 0}$).

The proof is similar to the proof of (2c).

Since $a < b$ there exists $x \in \mathbb{R}_{>0}$ such that
$$a + x = b.$$

Let $a = a_\ell \left(\frac{1}{10}\right)^\ell + a_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots$

$x = x_m \left(\frac{1}{10}\right)^m + x_{m+1}\left(\frac{1}{10}\right)^{m+1} + \cdots$

with $a_\ell \neq 0$ and $x_m \neq 0$ so that $a_\ell, x_m \in \{1, \ldots, 9\}$.

Let
$$a_{\leq k} = \begin{cases} a_\ell \left(\frac{1}{10}\right)^\ell + a_{\ell+1}\left(\frac{1}{10}\right)^{\ell+1} + \cdots + a_k \left(\frac{1}{10}\right)^k, & \text{if } k \geq \ell, \\ 0, & \text{if } k < \ell \end{cases}$$

Let $c = a_{\leq (m+2)} + \sqrt{2}\left(\frac{1}{10}\right)^{m+1}$.

Since $a_{\leq (m+2)} \in \mathbb{Q}_{\geq 0}$ and $\left(\frac{1}{10}\right)^{m+1} \in \mathbb{Q}_{\geq 0}$ and

$\sqrt{2} \notin \mathbb{Q}_{\geq 0}$ then $c = a_{\leq (m+2)} + \sqrt{2}\left(\frac{1}{10}\right)^{m+1} \notin \mathbb{Q}_{\geq 0}$.

To show: (da) $b > c$.

(db) $c > a$.

(da) To show: $b > c$.

$b = a + x = a + x_m \left(\frac{1}{10}\right)^m + \ldots$

$\geq a + \left(\frac{1}{10}\right)^m$, since $x_m \geq 1$

$> a + \sqrt{2}\left(\frac{1}{10}\right)^{m+1}$, since $\sqrt{2}\left(\frac{1}{10}\right)^{m+1} < 10 \cdot \left(\frac{1}{10}\right)^{m+1}$,

$\geq a_{\leq(m+2)} + \sqrt{2}\left(\frac{1}{10}\right)^{m+1}$

$= c$.

So $b > c$.

(db) To show: $c > a$

$c = a_{\leq(m+2)} + \sqrt{2}\left(\frac{1}{10}\right)^{m+1} > a_{\leq(m+2)} + \left(\frac{1}{10}\right)^{m+1}$

$> a_{\leq(m+2)} + \left(\frac{1}{10}\right)^{m+2}$

$= a_\ell \left(\frac{1}{10}\right)^\ell + \ldots + a_{m+2}\left(\frac{1}{10}\right)^{m+2} + \left(\frac{1}{10}\right)^{m+2}$

$\geq a_\ell \left(\frac{1}{10}\right)^\ell + \ldots + a_{m+2}\left(\frac{1}{10}\right)^{m+2} + a_{m+1}\left(\frac{1}{10}\right)^{m+1} + \ldots$

$= a$.

So $c > a$.

So $b > c > a$, with $c = a_{\leq(m+2)} + \sqrt{2}\left(\frac{1}{10}\right)^{m+1} \notin \mathbb{Q}_{\geq 0}$.