Assignment 2

MAST30026 Metric and Hilbert Spaces
Semester II 2016
Lecturer: Arun Ram
to be turned in before 10am on 13 October 2016

(1) (Eigenvectors)

(a) Carefully define linear operator, eigenvector and eigenvalue.

(b) Let V be a complex vector space and let $T\colon V \to V$ be a linear operator. Prove that there exists $v \in V$ with $v \neq 0$ such that v is an eigenvector of T.

(c) Use the proof of (b) to explicitly produce an eigenvector of the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 6 & 0 & 2 \\ \pi & \sqrt{7} & 0 \end{pmatrix}.$$

(d) Let V be the real vector space $\mathbb{R}^2$. Give an example of a linear transformation $T\colon V \to V$ that does not have a nonzero eigenvector.

(2) (Self adjoint operators) Let $T\colon V \to V$ be a self adjoint linear operator.

(a) Let v be an eigenvector of T with eigenvalue λ. Prove that $\lambda \in \mathbb{R}$.

(b) Let λ and γ be eigenvalues of T with $\lambda \neq \gamma$. Let

$$X_\lambda = \{v \in V \mid Tv = \lambda v\} \qquad \text{and} \qquad X_\gamma = \{v \in V \mid Tv = \gamma v\}.$$

Prove that X_λ is orthogonal to X_γ.

(3) (Alternative formula for the norm of a bounded self adjoint operator)

(a) Carefully define the norm of a linear operator $T\colon V \to V$.

(b) Carefully define bounded linear operator and self adjoint linear operator.

(c) Let $T\colon V \to V$ be a bounded self adjont linear operator. Prove that

$$\|T\| = \sup\{|\langle Tx, x\rangle| \mid \|x\| = 1\}.$$

(4) (Existence of eigenvectors of bounded self adjoint linear operators) Let H be a Hilbert space and let $T\colon H \to H$ be a bounded self adjoint operator.

(a) Show that there exists $x \in H$ with $\|x\| = 1$ and $|\langle Tx, x\rangle| = \|T\|$.

(b) Let $x \in H$ be as in (a). Show that x is an eigenvector of T with eigenvalue $\|T\|$.

(c) Use the proof of (a) to explicitly produce an eigenvector of the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ corresponding to the matrix

$$A = \begin{pmatrix} 1 & 5 & -2 \\ 5 & 0 & \pi \\ -2 & \pi & 0 \end{pmatrix}.$$

(5) (Compact linear operators) Let H be a Hilbert space.

(a) Carefully define compact linear operator.

(b) Give an example (with proof) of a bounded linear operator $T\colon \ell^2 \to \ell^2$ which is compact and a bounded linear operator $S\colon \ell^2 \to \ell^2$ which is not compact.

(c) Let $T\colon H \to H$ be a compact linear operator. Assume $\lambda \in \mathbb{C}$ and $\lambda \neq 0$ and let

$$X_\lambda = \{v \in H \mid Tv = \lambda v\}.$$

Show that X_λ is a subspace of H and that $\dim(X_\lambda)$ is finite.

(6) (Expansions in orthonormal sequences) Let H be a Hilbert space. Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H.

(a) Let $x \in H$. Show that $\displaystyle\sum_{n=1}^{\infty} |\langle x, a_n\rangle|^2 \leqslant \|x\|^2$.

(b) Show that $P(x) = \displaystyle\sum_{n=1}^{\infty} \langle x, a_n\rangle a_n$, exists in H.

(c) Let $W = \mathrm{span}\{a_1, a_2, \ldots\}$. With $P(x)$ as in (b), show that $P(x) \in \overline{W}$.

(d) With W as in (c) and $P(x)$ as in (b), show that $x - P(x) \in \overline{W}^{\perp}$.

(7) (Fourier's orthonormal sequences)

(a) Let $L^2(\mathbb{T})$ be the set of (Lebesgue measurable) functions $f\colon [0, 2\pi] \to \mathbb{C}$ such that

$$\int_0^{2\pi} |f(x)|^2 dx \ < \infty.$$

Prove that $L^2(\mathbb{T})$ with $\langle , \rangle \colon L^2(\mathbb{T}) \to L^2(\mathbb{T})$ given by

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)\overline{g(x)} dx,$$

is a Hilbert space.

(b) Prove that setting $a_n = e^{inx}$ defines an orthonormal sequence $(a_0, a_1, a_2, \ldots)$ in $L^2(\mathbb{T})$.

(c) Expand the function $f(x) = x^2$ in terms of the orthonormal sequence of (b).

(d) Evaluate the expansion in (c) at $x = 2\pi$ to prove that

$$\zeta(2) = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}.$$