<u>Tutorial:</u> Let $(X,d)$ be a metric space

A sequence $(x_1,x_2,\ldots)$ <u>converges to $x$</u> if $(x_1,x_2,\ldots)$ satisfies

if $\varepsilon\in\mathbb{R}_{>0}$ then there exists $N\in\mathbb{Z}_{>0}$ such that if $n\in\mathbb{Z}_{\geq N}$ then $d(x_n,x)<\varepsilon$.

Write
$$\lim_{n\to\infty} x_n = x \quad \text{if } (x_1,x_2,\ldots) \text{ converges to } x.$$

A sequence $(x_1,x_2,\ldots)$ is <u>Cauchy</u> if $(x_1,x_2,\ldots)$ satisfies

if $\varepsilon\in\mathbb{R}_{>0}$ then there exists $N\in\mathbb{Z}_{>0}$ such that if $m,n\in\mathbb{Z}_{>N}$ then $d(x_m,x_n)<\varepsilon$.

The metric space $(X,d)$ is <u>complete</u> if every Cauchy sequence in $X$ converges.

<u>Favorite example:</u> $\mathbb{Q}$ is not complete

$\mathbb{R}$ is complete

(with metric $d(x,y)=|x-y|$).

## Open and closed sets

The open sets in $\mathbb{R}$ are unions of open intervals.

The closed sets in $\mathbb{R}$ are complements of open sets

Give examples of

(a) an open set which also closed.

(b) an open set which is not closed

(c) a closed set which is not open

(d) a set which is not open and not closed.

The open sets in $[0,1]$ are

$U \cap [0,1]$, where $U$ is open in $\mathbb{R}$.

Give an example of

(a) an open set in $[0,1]$ which is not open in $\mathbb{R}$

(b) a closed set in $(0,1)$ which is not closed in $\mathbb{R}$.

## Topological and uniform spaces

A topological space is a set $X$ with each ~~collection~~ subset marked "open" or "not open" such that

(a) $\emptyset$ and $X$ are open

(b) unions of open sets are open

(c) finite intersections of open sets are open.

What are the topological spaces with $X=\{0,1\}$?

What are the uniform spaces with $X=\{0,1\}$?

A topological space is <u>Hausdorff</u> if $(X,\mathcal{T})$ satisfies

if $x,y\in X$ and $x\neq y$ then there exist

$U,V\in\mathcal{T}$ with $U\neq\emptyset$, $V\neq\emptyset$,

$x\in U$, $y\in V$ and $U\cap V=\emptyset$.

~~A topological space is /~~

<u>The "shape" of $\mathbb{R}$</u>: connectedness and compactness.

<u>Theorem</u> A subset $E\subseteq\mathbb{R}$ is <u>connected</u> if and only if $E$ is an interval.

<u>Theorem</u> A subset $E\subseteq\mathbb{R}$ is <u>compact</u> if and only if $E$ is closed and bounded.

Let $(X,\mathcal{T})$ be a topological space.
A subset $E\subseteq X$ is <u>connected</u> if there do not exist open sets $U$ and $V$ in $X$ with

$U\cap E\neq\emptyset$, $V\cap E\neq\emptyset$,

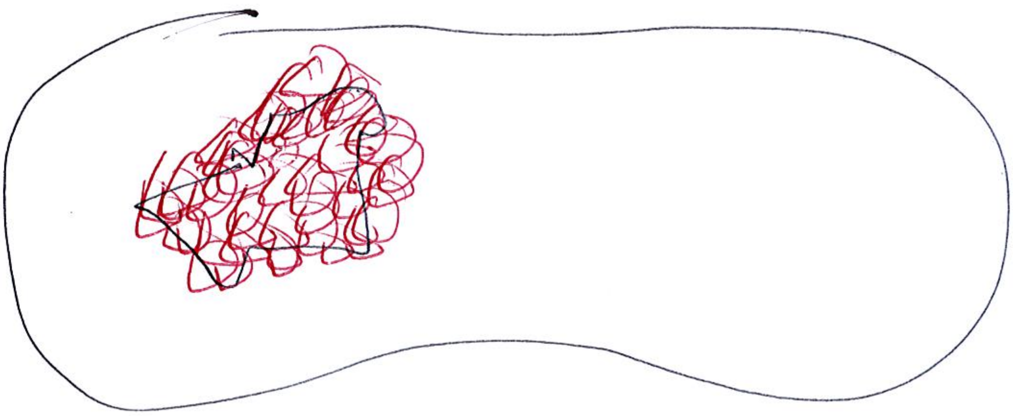
$E\subseteq U\cup V$ and $(U\cap E)\cap(V\cap E)=\emptyset$.

A subset $E \subseteq X$ is <u>compact</u> if every open cover of $E$ has a finite subcover.

In math:

If $\mathcal{S} \subseteq \mathcal{T}$ and $E \subseteq \bigcup_{U \in \mathcal{S}} U$

then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, U_2, \ldots, U_\ell \in \mathcal{S}$ such that $E \subseteq (U_1 \cup U_2 \cup \cdots \cup U_\ell)$.



For example: In a metric space,

$\mathcal{S} = \{ B_{\frac{1}{2}}(y) \mid y \in E \}$ is an open cover of $E$. ($\mathcal{S}$ is not finite unless $E$ is).

The "width of $E$" is finite only if $E$ is bounded.