Metric and Hilbert Spaces, Lecture 7, 11 August 2015 (1)
Univ. of Melbourne

Subspaces

Let $(X, \mathcal{T})$ be a topological space. Let $Y \subseteq X$.
The <u>subspace</u> <u>topology</u> on Y is

$$\mathcal{V} = \{ U \cap Y \mid U \in \mathcal{T} \}$$

<u>HW</u>: Show that the subspace topology on Y is a topology on Y.

Let (X, d) be a metric space. Let $Y \subseteq X$.
The <u>subspace metric on Y</u> is the function
$\rho : Y \times Y \to \mathbb{R}_{\geq 0}$ given by $\rho(y_1, y_2) = d(y_1, y_2)$

<u>HW</u>: Show that the subspace metric on Y is a metric on Y.

Products

Let (X, d_X) and (Y, d_Y) be metric spaces.
The <u>product metric space</u> is the set
$X \times Y$ with $d : (X \times Y) \times (X \times Y) \to \mathbb{R}_{\geq 0}$
given by

$$d((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

The product topology on $X \times Y$ is

$$\mathcal{T} = \{\text{unions of } A \times B \text{ where } A \in \mathcal{T}_X \text{ and } B \in \mathcal{T}_Y\}.$$

Filters

Let X be a set.

A filter on X is a collection $\mathcal{F}$ of subsets of X such that

(a) $\emptyset \notin \mathcal{F}$

(b) Finite intersections of sets in $\mathcal{F}$ are in $\mathcal{F}$.

(c) If $N \in \mathcal{F}$ and $E \subseteq X$ and $E \supseteq N$ then $E \in \mathcal{F}$.

Neighborhoods

Let $(X, \mathcal{T})$ be a topological space. Let $x \in X$.

A neighborhood of x is a subset N of X such that

there exists $U \in \mathcal{T}$ such that $x \in U$ and $U \subseteq N$.

The neighborhood filter of x is

$$\mathcal{N}(x) = \{\text{neighborhoods of } x\}.$$

Interiors and closures

Let $(X, \mathcal{T})$ be a topological space.

An open set in X is $U \in \mathcal{T}$.

A closed set in X is C with $C^c \in \mathcal{T}$.

Let $E \subseteq X$.

The interior of E is the subset E° of X such that

(a) E° is open in X and $E^\circ \subseteq E$,

(b) If U is open in X and $U \subseteq E$ then $U \subseteq E^\circ$.

In English: E° is the largest open set contained in E.

The closure of E is the subset $\bar{E}$ of X such that

(a) $\bar{E}$ is closed in X and $\bar{E} \supseteq E$,

(b) If C is closed in X and $C \supseteq E$ then $C \supseteq \bar{E}$.

In English: $\bar{E}$ is the smallest closed set containing E.

Let

$$\mathcal{N}(x) = \{\text{neighborhoods of } x\}, \quad \text{for } x \in X.$$

An <u>interior point of E</u> is an element $x \in X$ such that

there exists $N \in \mathcal{N}(x)$ such that $N \subseteq E$.

A <u>close point of E</u> is an element $x \in X$ such that

if $N \in \mathcal{N}(x)$ then $N \cap E \neq \emptyset$.

<u>Theorem</u> Let $(X, \mathcal{T})$ be a topological space and let $E \subseteq X$.

(a) The interior of E is the set of interior points of E.

(b) The closure of E is the set of close points of E.

<u>Proof of (a)</u> To show: $E^\circ = \{\text{interior points of } E\}$.

Let $I = \{\text{interior points of } E\}$.

To show: $E^\circ = I$.

To show: (aa) $I \subseteq E^\circ$

(ab) $E^\circ \subseteq I$.

(aa) Let $x \in I$

Then there exists $N \in \mathcal{N}(x)$ with $N \subseteq E$.

So there exists $U \in \mathcal{T}$ with $x \in U \subseteq N \subseteq E$.

Since $U \subseteq E$ and U is open, $U \subseteq E^\circ$.

So $x \in E^\circ$.

So $I \subseteq E^\circ$.

(ab) To show: $E^\circ \subseteq I$.

Let $x \in E^\circ$

Then E° is open and $x \in E^\circ \subseteq E$.

So x is an interior point of E.

So $x \in I$.

So $E^\circ \subseteq I$.

So $E^\circ = I$. //

Let $(X, \mathcal{T})$ be a topological space. Let $E \subseteq X$.

The <u>boundary</u> of A is $\partial A = \overline{A} \cap \overline{A^c}$.

The set A is <u>dense in X</u> if $\overline{A} = X$.

The set A is <u>nowhere dense</u> in X if $(\overline{A})^\circ = \emptyset$.