Number systems

$\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ with

$$(1+1+1) + (1+1+1+1) = 1+1+1+1+1+1+1$$

Prove: If $f: \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$ and $f(x+y) = f(x) + f(y)$ then f is "multiplication by k" where $k = f(1)$.

Real numbers

$$\mathbb{Z}_{\geq 0} = \left\{ a_\ell 10^\ell + a_{\ell-1} 10^{\ell-1} + \dots + a_1 10 + a_0 \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \{0, 1, \dots, 9\} \end{array} \right\}$$

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$$\mathbb{R} = \left\{ \pm(a_\ell 10^\ell + a_{\ell-1} 10^{\ell-1} + \dots) \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \{0, 1, \dots, 9\} \end{array} \right\}$$

with

$$x = y \quad \text{if} \quad \lim_{k \to \infty} |x_{\leq k} - y_{\leq k}| = 0$$

where

$$x_{\leq k} = x_\ell 10^\ell + x_{\ell-1} 10^{\ell-1} + \dots + x_k 10^k \quad \text{if} \quad x = x_\ell 10^\ell + x_{\ell-1} 10^{\ell-1} + \dots$$

POINT: $.9999\ldots = 1.000\ldots$

Note
$$\begin{aligned} .999\ldots &= 9 \cdot 10^{-1} + 9 \cdot 10^{-2} + \dots \\ &= \frac{9}{10}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \dots\right) \\ &= \frac{9}{10} \frac{1}{1 - \frac{1}{10}} = \frac{9}{10} \frac{10}{9} = 1 \\ &= 1.0000\ldots \end{aligned}$$

Extended polynomials

$$\mathbb{R}[t] = \left\{ a_0 t^0 + a_1 t^1 + \cdots \;\middle|\; \begin{array}{l} a_j \in \mathbb{R}, \text{ all but a finite} \\ \text{number of } a_j \text{ are } 0 \end{array} \right\}$$

$$= \left\{ a_\ell t^\ell + a_{\ell-1} t^{\ell-1} + \cdots + a_1 t + a_0 \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{\geq 0} \\ a_i \in \mathbb{R} \end{array} \right\}$$

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$$\mathbb{R}[[t]] = \{ a_0 t^0 + a_1 t^1 + a_2 t^2 + \cdots \mid a_i \in \mathbb{R} \}$$

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$$\mathbb{R}((t)) = \{ a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots \mid \ell \in \mathbb{Z}, a_i \in \mathbb{R} \}$$

Note: $\mathbb{R}(t) = \left\{ \dfrac{a(t)}{b(t)} \;\middle|\; \begin{array}{l} a(t), b(t) \in \mathbb{R}[t] \\ \text{and } b(t) \neq 0 \end{array} \right\}$

with $\dfrac{a(t)}{b(t)} = \dfrac{c(t)}{d(t)}$ if $a(t)d(t) = b(t)c(t)$

in the same way that

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z}, b \neq 0 \right\}$$

with $\dfrac{a}{b} = \dfrac{c}{d}$ if $ad = bc$.

Define
$$\begin{aligned} \mathrm{val}_t : \mathbb{R}((t)) &\to \mathbb{Z} \cup \{\infty\} \\ 0 &\mapsto \infty \\ a_{-\ell} t^{-\ell} + a_{-\ell+1} t^{-\ell+1} + \cdots &\mapsto -\ell \end{aligned}$$

so that $\mathrm{val}_t(a(t))$ is the exponent of the lowest power of t with nonzero coefficient.

Define $|\,|: \mathbb{R}((t)) \to \mathbb{R}$ by

$$|a(t)|_t = e^{-\mathrm{val}_t(a(t))}.$$

Define a metric $d: \mathbb{R}((t)) \times \mathbb{R}((t)) \to \mathbb{R}_{\geq 0}$ by

$$d(a(t), b(t)) = |a(t) - b(t)|_t.$$

<u>The p-adic numbers</u>

$$\mathbb{Z}_{\geq 0} = \left\{ a_0p^0 + a_1p^1 + a_2p^2 + \cdots \;\middle|\; \begin{array}{l} a_j \in \{0,1,\ldots,p-1\} \\ \text{all but a finite number of the} \\ a_j \text{ are } 0 \end{array} \right\}$$

$$= \{ a_0p^0 + a_1p^1 + \cdots + a_kp^k \mid k \in \mathbb{Z}_{\geq 0},\ a_i \in \{0,1,\ldots,p-1\} \}$$

$$\cap|$$

$$\mathbb{Z}_p = \{ a_0p^0 + a_1p^1 + \cdots \mid a_i \in \{0,1,\ldots,p-1\} \}$$

$$\cap|$$

$$\mathbb{Q}_p = \{ a_{-\ell}p^{-\ell} + a_{-\ell+1}p^{-\ell+1} + \cdots \mid a_i \in \{0,1,\ldots,p-1\},\ \ell \in \mathbb{Z} \}.$$

Define $\mathrm{val}_p : \mathbb{Q}_p \to \mathbb{Z} \cup \{\infty\}$ by

$$\mathrm{val}_p(a_{-\ell}p^{-\ell} + a_{-\ell+1}p^{-\ell+1} + \cdots) = -\ell$$

so that $\mathrm{val}_p(a)$ is the exponent of lowest t power of t with nonzero coefficient.

Define $|\,|_p : \mathbb{Q}_p \to \mathbb{R}$ by

$$|a|_p = e^{-\mathrm{val}_p(a)}$$

Define a metric $d: \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{R}_{\geq 0}$ by

$$d(a,b) = |a-b|_p.$$

Examples

In $\mathbb{R}$: $\frac{1}{2}, -1, \pi, \sqrt{2}$ all have decimal expansions

In $\mathbb{Q}_7$: $\frac{1}{2}, -1, -\frac{1}{6}, +\frac{1}{8}, 888$ all have 7-adic expansions.

In $\mathbb{R}((t))$: $\frac{1}{1-t}, e^t, \sin t, \frac{1}{t^3(1-t)}$ all have t-adic expansions.

$\mathbb{R}$, $\mathbb{Q}_7$ and $\mathbb{R}((t))$ are all metric spaces!

$\mathbb{R}$, $\mathbb{Q}_7$ and $\mathbb{R}((t))$ are all topological spaces!