Lecture 4: Metric and Hilbert Spaces 4 August 2015 (1)

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Function spaces:

(1) $\mathbb{R}^n = \{ x = (x_1, x_2, \ldots x_n) \mid x_1, x_2, \ldots x_n \in \mathbb{R} \}$

$= \{ x : \{1, 2, \ldots, n\} \to \mathbb{R} \}$

$= \{$functions from $\{1, 2, \ldots, n\}$ to $\mathbb{R}\}$.

Possible norms on $\mathbb{R}^n$:

$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ (standard norm)

$\|x\| = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$

$\|x\|_\infty = \sup \{ |x_1|, |x_2|, \ldots, |x_n| \}$

(2) $\mathbb{R}^\infty = \{ x = (x_1, x_2, \ldots) \mid x_i \in \mathbb{R} \}$

$= \{$sequences $(x_1, x_2, \ldots)$ in $\mathbb{R}\}$

$= \{ x : \mathbb{Z}_{>0} \to \mathbb{R} \}$

$= \{$functions from $\{1, 2, \ldots\}$ to $\mathbb{R}\}$

Possible norms on $\mathbb{R}^\infty$:

$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2 \right)^{1/2}$ gives ℓ^2

$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p \right)^{1/p}$ gives ℓ^p

$\|x\|_\infty = \sup \{ |x_1|, |x_2|, \ldots \}$ gives ℓ^∞

More function spaces

(3) $F = \{\text{functions } f : [0,1] \to \mathbb{R}\}$ or

$F = \{\text{functions } f : X \to \mathbb{R}\}$

with $\|f\|_\infty = \sup\{|f(x)| \mid x \in X\}$.

(4) Let $(V, \| \|)$ and $(W, \| \|)$ be normed vector spaces. Let

$F = \{\text{linear transformations } T : V \to W\}$

with $\|T\| = \sup\left\{ \frac{\|Tv\|}{\|v\|} \mid v \in V \right\}$

(5) Let (X, d) and (Y, ρ) be metric spaces. Let

$F = \{\text{functions } f : X \to Y\}$

with

$d_\infty : F \times F \to \mathbb{R}_{\geq 0} \cup \{0\}$ given by

$d_\infty(f_1, f_2) = \sup\{\rho(f(x), g(x)) \mid x \in X\}$.

<u>Remark on (4)</u>: If $(V, \| \|)$ and $(W, \| \|)$ are normed vector spaces, then

$\mathcal{B}(V, W) = \{\text{linear transformations } T : V \to W \mid \|T\| < \infty\}$

is a normed vector space.

$\mathbb{R}^2 = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{R}\}$

$\mathbb{R}^2$ is an $\mathbb{R}$-vector space:

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, y_1 + y_2)$ and $c(x_1, x_2) = (cx_1, cx_2)$

Define

$\langle\, , \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + x_2 y_2$.

and $\| \; \| : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ by $\|x\|^2 = \langle x, x \rangle$ so that

$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\langle (x_1, x_2), (x_1, x_2) \rangle}$

$= \sqrt{x_1^2 + x_2^2}$, if $x = (x_1, x_2) \in \mathbb{R}^2$

Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ by

$d(x, y) = \|y - x\|$ so that

$d((x_1, x_2), (y_1, y_2)) = \|(y_1, y_2) - (x_1, x_2)\| = \|(y_1 - x_1, y_2 - x_2)\|$

$= \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2}$

So $\mathbb{R}^2$ is a pos. def. symm. inner product space,

and a normed vector space

and a metric space.

An ε-slice for $\mathbb{R}^2$ is

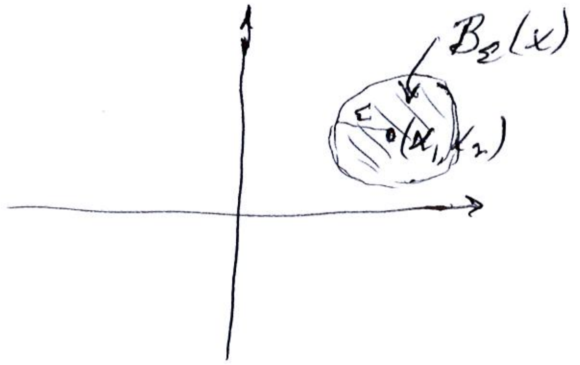
$$B_\varepsilon = \{(x,y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid d(x,y) < \varepsilon\}$$
$$= \{((x_1,x_2),(y_1,y_2)) \mid d((x_1,x_2),(y_1,y_2)) < \varepsilon\}$$
$$= \{((x_1,x_2),(y_1,y_2)) \mid \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2} < \varepsilon\}.$$

The uniformity on $\mathbb{R}^2$ is

$$\mathcal{X} = \{\text{subsets of } \mathbb{R}^2 \times \mathbb{R}^2 \text{ that contain a } B_\varepsilon\}.$$

The open ball of radius ε at x is

$$B_\varepsilon(x) = \{y \in \mathbb{R}^2 \mid d(y,x) < \varepsilon\}$$



The topology on $\mathbb{R}^2$ is

$$\mathcal{T} = \{\text{unions of } B_\varepsilon(x)\}$$

An open set in $\mathbb{R}^2$ is a union of open balls.

A closed set in $\mathbb{R}^2$ is the complement of an open set in $\mathbb{R}^2$.