Quantum Mechanics (Reference: Schiff Chapter 6)

The Hilbert space is

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \mid \|f\| < \infty \}$$ with

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)} \, dx$$

Operators

$x$ = position operator

$p$ = momentum operator

$E = H$ = Hamiltonian
= energy operator

$a^\dagger$ = creation operator

$a$ = annihilation operator

Constants

$m$ = mass of particle

$K$ = spring constant

$\omega_c$ = angular frequency of harmonic oscillator

$\hbar$ = Planck's constant.

The quantum condition

$$xp - px = i\hbar$$

The harmonic oscillator

Remark $F = -\frac{d}{dx} V$ or $F = -\nabla V$

and $F = -Kx$ corresponds to $V(x) = \frac{1}{2} K x^2$

$F$ is force and $V(x)$ is potential energy.

## Equations

$$\omega_c = \left(\frac{K}{m}\right)^{\frac{1}{2}}, \quad p = -i\hbar\frac{\partial}{\partial x} = -i\hbar\nabla, \quad E = i\hbar\frac{\partial}{\partial t}$$

$$H = \frac{p^2}{2m} + \frac{1}{2}Kx^2 \quad \text{and} \quad H = \left(a^\dagger a + \tfrac{1}{2}\right)\hbar\omega_c$$

$$aa^\dagger = \frac{H}{\hbar\omega_c} + \frac{1}{2}, \quad a^\dagger a = \frac{H}{\hbar\omega_c} - \frac{1}{2}, \quad aa^\dagger - a^\dagger a = 1$$

$$xp - px = i\hbar, \quad xH - Hx = \frac{i\hbar}{m}p, \quad pH - Hp = -i\hbar Kx$$

$$x = \left(\frac{\hbar}{2m\omega_c}\right)^{\frac{1}{2}}(a^\dagger + a), \quad p = i\left(\frac{m\hbar\omega_c}{2}\right)^{\frac{1}{2}}(a^\dagger - a)$$

$$a^\dagger = \frac{-i}{(2m\hbar\omega_c)^{\frac{1}{2}}}(p + im\omega_c x), \quad a = \frac{i}{(2m\hbar\omega_c)^{\frac{1}{2}}}(p - im\omega_c x)$$

## Matrices

$$a = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2^{\frac{1}{2}} & & \\ & & 0 & 3^{\frac{1}{2}} & \\ & & & 0 & \\ & & & & \ddots \end{pmatrix} \qquad a^\dagger = \begin{pmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2^{\frac{1}{2}} & 0 & & \\ & & 3^{\frac{1}{2}} & 0 & \\ & & & & \ddots \end{pmatrix}$$

$$H = \begin{pmatrix} \frac{1}{2}\hbar\omega_c & & & \\ & \frac{3}{2}\hbar\omega_c & & \\ & & \frac{5}{2}\hbar\omega_c & \\ & & & \ddots \end{pmatrix}$$

$\frac{1}{2}\hbar\omega_c$ is the zero point energy.

with respect to the basis

$$\{|0\rangle, |1\rangle, |2\rangle, \dots\} \text{ of } L^2(\mathbb{R})$$

given by

$$|0\rangle = e^{-ix^2}, \quad |1\rangle = a|0\rangle, \quad |2\rangle = \frac{1}{\sqrt{2}} a|1\rangle, \quad |3\rangle = \frac{1}{\sqrt{3}} a|2\rangle, \dots$$

## The Schrödinger equation

$$H u_K = E_K u_K$$

$u_K$ are energy eigenfunctions

$E_K$ are energy eigenvalues

Alternatively,

$$i\hbar \frac{\partial}{\partial t} u_K = \left( \frac{-\hbar^2}{2m} \nabla^2 + V(x) \right) u_K = E_K u_K .$$

## The Schrödinger representation

Let, for $s, t \in \mathbb{R}$,

$$U(s) = e^{isp} \quad \text{and} \quad V(t) = e^{itx}$$

Then

$$U(s)V(t) = e^{-ist} V(s)U(t)$$

and, for $f: \mathbb{R} \to \mathbb{C}$ in $L^2(\mathbb{R})$,

$$U(s)f(x) = f(x+s) \quad \text{and} \quad V(t)f(x) = e^{ixt} f(x).$$

(see Wikipedia, Oscillator representation)

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} \mid \|f\|_2 < \infty \}$$

where

$$\langle f, g \rangle = \int_{\mathbb{R}} f(x)\, \overline{g(x)}\, dx$$

<u>Check</u>

$$PQ - QP = \left( -i \frac{d}{dx} x + x \left( -i \frac{d}{dx} \right) \right) f$$

$$= -i \frac{d}{dx}(xf) - i \left( x \frac{df}{dx} \right)$$

$$= -i \left( x \frac{df}{dx} + f \right) - i x \frac{df}{dx} = -if.$$

So $PQ - QP = -i$

<u>Note:</u>

$$a^+ a = \frac{1}{2}(P + iQ)(P - iQ)$$

$$= \frac{1}{2}\left( P^2 - iPQ + iQP + Q^2 \right)$$

$$= \frac{1}{2}\left( (-i)^2 \frac{d^2}{dx^2} - i(PQ - QP) + x^2 \right)$$

$$= \frac{1}{2}\left( -\frac{d^2}{dx^2} - i(-i) + x^2 \right)$$

$$= \frac{1}{2}\left( -\frac{d^2}{dx^2} - 1 + x^2 \right)$$

So

$$a^+ a + \frac{1}{2} = -\frac{d^2}{dx^2} + x^2$$