Metric & Hilbert, Lecture 33, 15 October 2015 ①

Theorem 13.3 Let H be a Hilbert space. Let $T: H \to H$ be a bounded selfadjoint operator.

Let
$$m = \inf\{\langle Tu, u\rangle \mid \|u\| = 1\}$$
$$M = \sup\{\langle Tu, u\rangle \mid \|u\| = 1\}$$

Then $\|T\| = \max\{-m, M\} = \sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\}$.

Proof Assume $|m| \leq M$ (otherwise replace T by $-T$).

To show: (a) $\|T\| \leq M$

(b) $\|T\| \geq M$.

(a) So $M = \sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\}$.

Let $x \in H$ with $Tx \neq 0$ and $\|x\| = 1$.

Let $y = \dfrac{Tx}{\|Tx\|}$. Then

$$\|Tx\| = \frac{\langle Tx, Tx\rangle}{\|Tx\|} = \langle Tx, y\rangle = \mathrm{Re}\,\langle Tx, y\rangle$$
$$= \frac{1}{4}\left(4\,\mathrm{Re}\langle Tx, y\rangle\right) = \frac{1}{4}\left(2\langle Tx, y\rangle + 2\overline{\langle Tx, y\rangle}\right)$$
$$= \frac{1}{4}\left(2\langle Tx, y\rangle + 2\langle y, Tx\rangle\right)$$
$$= \frac{1}{4}\left(2\langle Tx, y\rangle + 2\langle Ty, x\rangle\right), \text{ since } T \text{ is selfadjoint}$$
$$= \frac{1}{4}\left(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right)$$
$$\leq \frac{1}{4}\left|\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\right|$$

$$\leq \frac{1}{4}\left(|\langle T(x+y), (x+y)\rangle| + |\langle T(x-y), x-y\rangle| \right)$$

$$= \frac{1}{4}\left(\left|\left\langle \frac{T(x+y)}{\|x+y\|}, \frac{x+y}{\|x+y\|}\right\rangle\right| \|x+y\|^2 + \left|\left\langle \frac{T(x-y)}{\|x-y\|}, \frac{x-y}{\|x-y\|}\right\rangle\right| \|x-y\|^2 \right)$$

$$\leq \frac{1}{4}\left(M\|x+y\|^2 + M\|x-y\|^2 \right)$$

$$= \frac{1}{4}M\left(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle \right)$$

$$= \frac{1}{4}M\left(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle \right)$$

$$= \frac{1}{4}M\left(2\|x\|^2 + 2\|y\|^2 + 2\,\mathrm{Re}(\langle x,y\rangle) - 2\,\mathrm{Re}(\langle x,y\rangle) \right)$$

$$= \frac{1}{2}M\left(\|x\|^2 + \|y\|^2 \right)$$

$$= \frac{1}{2}M(1+1) = M. \;//$$

<u>Theorem</u> <u>13.4</u> Let H be a Hilbert space. Let $T: H \to H$ be a bounded self adjoint compact linear operator. Let $M = \|T\| = \sup\{|\langle Tu, u\rangle| \mid \|u\| = 1\}$. Then there exists an eigenvector $y \in H$ with $Ty = My$.

<u>Proof</u> Let $u_1, u_2, \ldots \in H$ with $\|u_n\| = 1$ and

$$\lim_{n\to\infty} \langle Tu_n, u_n\rangle = M.$$

Then

$$\|Tu_n - Mu_n\|^2 = \langle Tu_n - Mu_n, Tu_n - Mu_n\rangle$$

$$= \langle Tu_n, Tu_n\rangle - \langle Tu_n, Mu_n\rangle - \langle Mu_n, Tu_n\rangle + M^2\langle u_n, u_n\rangle$$

$$= \|Tu_n\|^2 - 2M\langle Tu_n, u_n\rangle + M^2\|u_n\|^2$$

$$= \|Tu_n\|^2 - 2M\langle Tu_n, u_n\rangle + M^2$$

$$\le \|T\|^2 - 2M\langle Tu_n, u_n\rangle + M^2$$

$$= 2M^2 - 2M\langle Tu_n, u_n\rangle.$$

So $\lim_{n\to\infty} \|Tu_n - Mu_n\| = 0$

Since T is compact, $Tu_1, Tu_2, \ldots$ has a cluster point y.

There is a subsequence $Tu_{n_1}, Tu_{n_2}, \ldots$ with

$\lim_{k\to\infty} Tu_{n_k} = y$. So $\lim_{k\to\infty} \|Tu_{n_k} - y\| = 0$.

Then

$$\lim_{k\to\infty} \|y - Mu_{n_k}\| \le \lim_{k\to\infty} \|y - Tu_{n_k}\| + \|Tu_{n_k} - Mu_{n_k}\|$$

$$= 0 + 0 = 0.$$

So $\lim_{k\to\infty} Mu_{n_k} = y$.

Since $\|u_{n_k}\| = 1$ then $\|y\| = M$. So $y \neq 0$.

and $\|Ty - My\| = \lim_{k\to\infty} \|TMu_{n_k} - M\cdot Mu_{n_k}\| = 0$. So $Ty = My$. //