Metric & Hilbert Spaces, Lecture 32, 14.10.2015

Norm of a finite square matrix

Let $B \in M_n(\mathbb{C})$. Then let

$$A = B^* B \quad \text{where } B^* = \bar{B}^t$$

Then
$$A^* = (B^*B)^* = B^*(B^*)^* = B^*B = A$$
So A is self adjoint.

By a Theorem from 1st year (why is this true):

There exists $K \in M_n(\mathbb{C})$ with $K\bar{K}^t = 1$ and

$$KAK^{-1} = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Proposition (a) $\|A\| = \|KAK^{-1}\|$

(b) If γ is the largest eigenvalue of A then $\|B\| = \sqrt{\gamma}$.

Proof (1) If $x \in V$ then, since $K^*K = 1$ (K is unitary)

$$\|Kx\|^2 = \langle Kx, Kx\rangle = \langle x, K^*Kx\rangle = \langle x, x\rangle = \|x\|^2.$$

So $\|Kx\| = \|x\|$

(2) If $x \in V$ then

$$\|KAK^{-1}x\| = \|AK^{-1}x\| \leq \|A\|\,\|K^{-1}x\| = \|A\|\,\|x\|$$

So $\|KAK^{-1}\| \leq \|A\|$.

Since $\|K^{-1}(KAK^{-1})K\| \leq \|KAK^{-1}\|$ then

$$\|A\| \leq \|KAK^{-1}\|$$

So $\|KAK^{-1}\| = \|A\|$.

(3) If $x \in V$ and $Ax = \lambda x$ then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^*Bx\rangle = \langle x, \lambda x\rangle = \lambda \|x\|^2$$

Since $\|Bx\|^2 \in \mathbb{R}_{\geq 0}$ and $\|x\|^2 \in \mathbb{R}_{\geq 0}$ then $\lambda \in \mathbb{R}_{\geq 0}$

and $\|Bx\| = \sqrt{\lambda}\, \|x\|$. So $\|B\| \geq \sqrt{\lambda}$.

(4) If $\{a_1, a_2, \ldots, a_n\}$ is a basis of eigenvectors of A and

$$x = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$$

is a vector in V then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^*Bx\rangle$$

$$= \langle x, A(x_1 a_1 + x_2 a_2 + \cdots + x_n a_n)\rangle$$

$$= \langle x_1 a_1 + x_2 a_2 + \cdots + x_n a_n, x_1 \lambda_1 a_1 + \cdots + x_n \lambda_n a_n\rangle$$

$$= \lambda_1 |x_1|^2 + \lambda_2 |x_2|^2 + \cdots + \lambda_n |x_n|^2$$

$$\leq \max\{\lambda_1, \ldots, \lambda_n\} (|x_1|^2 + \cdots + |x_n|^2)$$

$$= \max\{\lambda_1, \ldots, \lambda_n\} \langle x, x\rangle$$

$$= \max\{\lambda_1, \ldots, \lambda_n\} \|x\|^2 =: \gamma \|x\|^2.$$

So $\|B\| \leq \sqrt{\gamma}$.

So $\|B\| = \sqrt{\gamma}$.

<u>Fredholm's Theorem</u> Let H be a Hilbert space and let $T: H \to H$ be a compact linear operator. Then

$\lambda - K$ is injective if and only if $\lambda - K$ is bijective.

<u>Theorem</u> <u>13.3</u> Let H be a Hilbert space and let $T: H \to H$ be a bounded self adjoint operator.

Let

$$m = \inf\{\langle Tu, u\rangle \mid \|u\| = 1\}$$

$$M = \sup\{\langle Tu, u\rangle \mid \|u\| = 1\}.$$

Then $\|T\| = \max\{-m, M\}$.

<u>Proof</u> Assume $|m| \leq M$ (otherwise replace T by $-T$).

To show: (a) $\|T\| \geq M$

(b) $\|T\| \leq M$.

(a) Assume $u \in H$ and $\|u\| = 1$. Then, by Cauchy-Schwarz,

$$|\langle Tu, u\rangle| \leq \|Tu\| \|u\| \leq \|T\| \|u\| \cdot \|u\| = \|T\|.$$

So $M \leq \|T\|$.

<u>Proposition</u> Let H be a Hilbert space and let $T : H \to H$ be a bounded self adjoint operator.

Let $M = \|T\|$. Then

$$M - \|T\| \text{ is not invertible.}$$

<u>Proof</u> Since $M = \sup \{ \langle Tu, u \rangle \mid \|u\| = 1 \}$, let $u_n \in H$ with $\|u_n\| = 1$ such that

$$\lim_{n \to \infty} \langle Tu_n, u_n \rangle = M.$$

Then

$$\|Tu_n - Mu_n\|^2 = \langle Tu_n - Mu_n, Tu_n - Mu_n \rangle$$

$$= \langle Tu_n, Tu_n \rangle - 2 \langle Tu_n, u_n \rangle M + M^2 \langle u_n, u_n \rangle$$

$$= \|Tu_n\|^2 - 2M \langle Tu_n, u_n \rangle + M^2$$

$$\leq \|T\|^2 - 2M \langle Tu_n, u_n \rangle + M^2$$

$$= 2M^2 - 2M \langle Tu_n, u_n \rangle .$$

Since the right handside approaches 0 as $n \to \infty$ then

$$\lim_{n \to \infty} \|Tu_n - Mu_n\|^2 = 0.$$

So $T - M$ is not invertible.

Theorem Let H be a Hilbert space and let $T: H \to H$ be a compact selfadjoint operator.
Let $M = \|T\|$. Then there exists an eigenvector y of T with eigenvalue M.

Proof Let $u_n \in H$ with $\|u_n\| = 1$ and

$$\lim_{n \to \infty} \langle Tu_n, u_n \rangle = M$$

Then

$$\lim_{n \to \infty} \| Tu_n - Mu_n \|^2 = 0.$$

Let y be a cluster point of $Tu_1, Tu_2, \ldots$

Then $Tu_{n_1}, Tu_{n_2}, \ldots$ converges to y, so

$$\lim_{k \to \infty} \|Tu_{n_k} - y\| = 0$$

Then

$$\lim_{k \to \infty} \|y - Mu_{n_k}\| \leq \lim_{k \to \infty} \|y - Tu_{n_k}\| + \|Tu_{n_k} - Mu_{n_k}\|$$

$$= 0 + 0 = 0.$$

So $\lim_{k \to \infty} Mu_{n_k} = y$.

So $\|y\| = M$ and $y \neq 0$

and $\|Ty - My\| = 0$ so that $Ty = My$.