Let V be a vector space and let

$T: V \longrightarrow V$ be a linear transformation.

(1) T has an eigenvector if and only if there exists $\lambda \in \mathbb{C}$ such that $\ker(\lambda - T) \neq 0$.

Proof.

$$\ker(\lambda - T) = \{v \in V \mid (\lambda - T)v = 0\}$$
$$= \{v \in V \mid Tv = \lambda v\}.$$

(2) $\ker(\lambda - T) = 0$ if and only if T is injective.

(3) Assume V is finite dimensional.

$\ker(\lambda - T) = 0$ if and only if T is bijective.

(4) Assume V is finite dimensional.

T is bijective if and only if $\det(A) \in \mathbb{C}^\times$, where A is the matrix of T with respect to a basis of V, and $\mathbb{C}^\times = \{c \in \mathbb{C} \mid c \neq 0\}$.

The characteristic polynomial of A is

$$\det(\lambda - A)$$

(a polynomial in the variable λ).

<u>Main theorem</u> If H is a separable Hilbert space and $T: H \to H$ is a bounded compact self adjoint operator then there exists an orthonormal basis of eigenvectors of H.

Let H be finite dimensional.

Let $\{b_1, b_2, \ldots, b_n\}$ be a basis of H.

Then use Gram-Schmidt to get an orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ of H.

Then there is another orthonormal basis of H, $\{a_1, a_2, \ldots, a_n\}$ with $Ta_r = \lambda_r a_r$ for $r \in \{1, \ldots, n\}$.

Let $K = (k_{ij})$ be the transition matrix between $\{e_1, e_2, \ldots, e_n\}$ and $\{a_1, a_2, \ldots, a_n\}$:

$$a_i = \sum_{j=1}^{n} k_{ji} e_j = Ke_i$$

Then

(a) K is unitary

(b) $KAK^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

where A is the matrix of T with respect to the basis $\{e_1, e_2, \ldots, e_n\}$. Recall $A = A^*$

Proof of (a): To show: $KK^* = 1$.

$$\delta_{ij} = \langle a_i, a_j \rangle = \langle Ke_i, Ke_j \rangle = \langle e_i, K^*Ke_j \rangle \;//.$$

Point: If $A \in M_n(\mathbb{C})$ and $A = \overline{A}^t$ then there exists $K \in M_n(\mathbb{C})$ with

$$K\overline{K}^t = 1, \quad \text{and} \quad KAK^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

Norm of a finite square matrix

Let $B \in M_n(\mathbb{C})$. Then let

$$A = B^*B \quad \text{where} \quad B^* = \overline{B}^t.$$

Then

$$A^* = (B^*B)^* = B^*(B^*)^* = B^*B = A.$$

So there exists $K \in M_n(\mathbb{C})$ with $K\overline{K}^t = 1$ and $KAK^{-1} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

To show: (a) $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}_{\geq 0}$.

(b) If $\gamma = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ then $\|A\| = \gamma$.

(c) If $\gamma = \max\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ then $\|B\| = \sqrt{\gamma}$.

Let $V = \mathbb{C}^n$.

(1) If $x \in V$ then

$$\|Kx\|^2 = \langle Kx, Kx\rangle = \langle x, K^*Kx\rangle = \langle x, x\rangle = \|x\|^2.$$

and since $\|Kx\|^2 \in \mathbb{R}_{\geq 0}$ and $\|x\|^2 \in \mathbb{R}_{\geq 0}$ then

$$\|Kx\| = \|x\|.$$

(2) If $x \in V$ then

$$\|KAK^{-1}x\| = \|AK^{-1}x\| \leq \|A\|\,\|K^{-1}x\| = \|A\|\,\|x\|.$$

So $\|KAK^{-1}\| \leq \|A\|$.

(3) Since $\|K^{-1}(KAK^{-1})K\| \leq \|KAK^{-1}\|$, then

$$\|A\| \leq \|KAK^{-1}\|.$$

So $\|A\| = \|KAK^{-1}\|$

(4) If $x \in V$ ~~then~~ and $Ax = \lambda x$ then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^*Bx\rangle = \langle x, \lambda x\rangle = \lambda\|x\|^2.$$

Since $\|Bx\|^2 \in \mathbb{R}_{\geq 0}$ and $\|x\|^2 \in \mathbb{R}_{\geq 0}$ then $\lambda \in \mathbb{R}_{\geq 0}$.
So $\|Bx\| = \sqrt{\lambda}\,\|x\|$ and $\|B\| \geq \sqrt{\lambda}$.

(5) If $x \in V$ and $x = \gamma_1 a_1 + \cdots + \gamma_n a_n$ then

$$\|Bx\|^2 = \langle Bx, Bx\rangle = \langle x, B^*Bx\rangle$$

$$= \langle x, A(\gamma_1 a_1 + \cdots + \gamma_n a_n)\rangle = \langle \gamma_1 a_1 + \cdots + \gamma_n a_n, \gamma_1\lambda_1 a_1 + \cdots + \gamma_n\lambda_n a_n\rangle$$

$$= \gamma_1^2\lambda_1 \cancel{a_1} + \cdots + \gamma_n^2\lambda_n \cancel{a_n} \leq \max\{\lambda_1, \ldots, \lambda_n\}(\gamma_1^2 + \cdots + \gamma_n^2)$$

$$\leq \max\{\lambda_1, \ldots, \lambda_n\}\langle x, x\rangle \leq \max\{\lambda_1, \ldots, \lambda_n\}\|x\|^2.$$