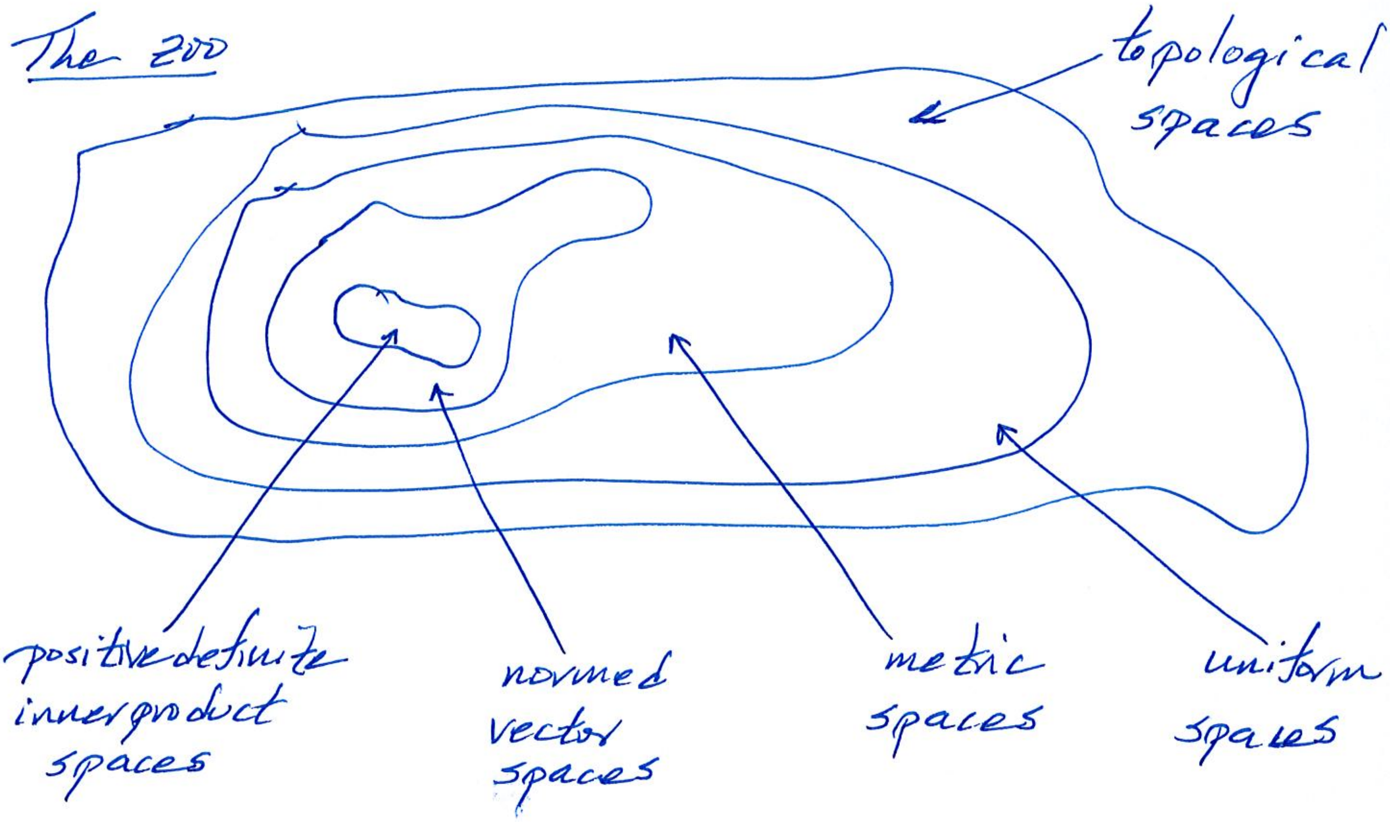
Lecture 2: Metric and Hilbert spaces 29 July 2015

The zoo



$$
\begin{array}{c}
\{\text{topological spaces}\} \\
\cup\!| \\
\{\text{uniform spaces}\} \\
\cup\!| \\
\{\text{metric spaces}\} \\
\cup\!| \\
\{\text{normed vector spaces}\} \\
\cup\!| \\
\left\{\begin{array}{c}\text{positive definite inner product} \\ \text{spaces}\end{array}\right\}
\end{array}
$$

Topological spaces and uniform spaces

Continuous functions are for comparing topological spaces.

Let $(X, \mathcal{T})$ and $(Y, \mathcal{V})$ be topological spaces.

An open set in X is $U \subseteq X$ such that $U \in \mathcal{T}$.

An open set in Y is $V \subseteq Y$ such that $V \in \mathcal{V}$.

A continuous function from $(X, \mathcal{T})$ to $(Y, \mathcal{V})$ is a function $f: X \to Y$ such that

if $V \in \mathcal{V}$ then $f^{-1}(V) \in \mathcal{T}$.

Recall: $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$

Uniformly continuous functions are for comparing uniform spaces.

Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be uniform spaces.

An entourage in X is $E \subseteq X \times X$ such that $E \in \mathcal{X}$

An entourage in Y is $G \subseteq Y \times Y$ such that $G \in \mathcal{Y}$

A uniformly continuous function from $(X, \mathcal{X})$ to $(Y, \mathcal{Y})$ is a function $f: X \to Y$ such that

if G is an entourage in Y then

$(f \times f)^{-1}(G)$ is an entourage in X.

Recall: $(f \times f)^{-1}(G) = \{(x_1, x_2) \in X \times X \mid (f(x_1), f(x_2)) \in G\}$.

Metric spaces

A *metric space* is a set X with a function

$$d : X \times X \to \mathbb{R}_{\geq 0} \qquad (x,y) \mapsto d(x,y) \qquad \text{such that}$$

(a) If $x, y \in X$ then $d(x,y) = d(y,x)$.

(b) If $x, y, z \in X$ then $d(x,y) \leq d(x,z) + d(z,y)$

(c) If $x \in X$ then $d(x,x) = 0$.

The *entourage slice of width* ε is

$$B_\varepsilon = \{(x,y) \in X \times X \mid d(x,y) < \varepsilon\}, \quad \text{for } \varepsilon \in \mathbb{R}_{>0}.$$

The *open ball of radius* ε *at* x is

$$B_\varepsilon(x) = \{y \in X \mid d(x,y) < \varepsilon\}, \quad \text{for } \varepsilon \in \mathbb{R}_{>0},\ x \in X.$$

The *metric space topology on* (X,d) is the smallest topology $\mathcal{T}$ on X which contains the sets

$$B_\varepsilon(x), \quad \text{for } \varepsilon \in \mathbb{R}_{>0} \text{ and } x \in X.$$

The *metric space uniformity on* (X,d) is the smallest uniformity $\mathcal{X}$ on X which contains the sets

$$B_\varepsilon, \quad \text{for } \varepsilon \in \mathbb{R}_{>0}$$

Problem: Does $\mathcal{T}$ exist? Does $\mathcal{X}$ exist?

POINT: If (X,d) is a metric space then $(X,\mathcal{T})$ is a topological space and $(X,\mathcal{X})$ is a uniform space.

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

A <u>positive definite inner product space</u> is a positive definite symmetric inner product space or a positive definite Hermitian inner product space.

A <u>positive definite symmetric inner product space</u> is a vector space V ~~over $\mathbb{K}$~~ with a function

$$\begin{array}{rcl} \langle\,,\,\rangle : V \times V & \to & \mathbb{K} \\ (v, w) & \mapsto & \langle v, w\rangle \end{array}$$
such that

(a) If $c_1, c_2 \in \mathbb{K}$ and $v_1, v_2 \in V$ and $w \in V$ then
$$\langle c_1 v_1 + c_2 v_2, w\rangle = c_1\langle v_1, w\rangle + c_2\langle v_2, w\rangle$$
(b) If $v, w \in V$ then $\langle w, v\rangle = \langle v, w\rangle$

(c) If $v \in V$ then $\langle v, v\rangle \in \mathbb{R}_{\geq 0}$

(d) If $v \in V$ and $\langle v, v\rangle = 0$ then $v = 0$.

A <u>normed</u> <u>vector space</u> is a vector space with a function
$$\begin{array}{rcl} \|\ \| : V & \to & \mathbb{R}_{\geq 0} \\ v & \mapsto & \|v\| \end{array}$$
such that

(a) If $c \in \mathbb{K}$ and $v \in V$ then $\|cv\| = |c| \cdot \|v\|$

(b) If $v_1, v_2 \in V$ then $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$

(c) If $v \in V$ and $\|v\| = 0$ then $v = 0$.