Writing $\langle v,w\rangle = \psi_v(w)$ gives a bijection

$$\left\{\begin{matrix}\text{bilinear forms}\\ \langle\,\rangle: V\times W\to\mathbb{C}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{linear transformations}\\ \text{from } V \text{ to } W^*\end{matrix}\right\}$$

$$\begin{matrix}\langle\,\rangle: V\times W\to\mathbb{C} & \longmapsto & \psi: V \to W^* \\ & & v \mapsto \psi_v: W\to\mathbb{F} \\ & & \qquad w\mapsto \langle v,w\rangle\end{matrix}$$

The orthogonal to W is

$$W^\perp = \ker\psi = \{v\in V \mid \text{if } w\in W \text{ then } \langle v,w\rangle = 0\}.$$

Theorem Let H be a Hilbert space and let $W\subseteq H$ be a subspace of H. If W is closed then

$$H = W\oplus W^\perp$$

(i.e. if $x\in H$ then there is a unique expansion $x = w_1+w_2$ with $w_1\in W$ and $w_2\in W^\perp$.)

Theorem Let H be Hilbert space. Then

$$H\xrightarrow{\psi} H^* \text{ is an isomorphism.}$$

Proposition Let H_1 and H_2 be Hilbert spaces. Let $T: H_1\to H_2$ be a bounded linear operator. Then $T^*: H_2\to H_1$ is given by

$$\langle Tx, y\rangle_2 = \langle x, T^*y\rangle_1, \quad \text{for } x\in H_1 \text{ and } y\in H_2.$$

Let H be a Hilbert space and let

$T: H \to H$ be a bounded linear operator.

(a) T is <u>self adjoint</u> if $T = T^*$

(b) T is <u>positive</u> if $T = T^*$ and

if $x \in H$ then $\langle Tx, x \rangle \in \mathbb{R}_{\geq 0}$

(c) T is <u>unitary</u> if $TT^* = T^*T = 1$.

(d) T is an <u>isometry</u> if T satisfies

if $x, y \in H$ then $\langle Tx, Ty \rangle_2 = \langle x, y \rangle_1$.

Let X be a normed vector space.

A bounded linear operator $T: X \to X$ is <u>compact</u>

if $\overline{\{Tx \mid \|x\| = 1\}}$ is compact

i.e. if $(x_1, x_2, \ldots)$ is a sequence in $\{x \in H \mid \|x\| = 1\}$

then there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$

such that

$(Tx_{n_1}, Tx_{n_2}, \ldots)$ converges.

In other words: $Tx_1, Tx_2, \ldots$ has a cluster point.

Theorem Let H be a Hilbert space and let

$T: H \to H$ be a bounded, selfadjoint linear operator.

Let

$$m = \inf\{\langle Tu, u\rangle \mid \|u\|=1\} \text{ and } M = \sup\{\langle Tu,u\rangle \mid \|u\|=1\}.$$

(i) If $\lambda \notin [m, M]$ then $\lambda I - T$ is ~~not~~ a bijection.

(ii) $mI - T$ and $M \cdot I - T$ are not bijections.

(iii) $\|T\| = \max\{-m, M\}$, i.e.

$$\|T\| = \sup\{\, |\langle Tu, u\rangle| \mid \|u\|=1\}.$$

Theorem Let H be a Hilbert space and let

$T: H \to H$ be a compact selfadjoint operator.

Then there exists an orthonormal basis of eigenvectors of H.

Note: If $u \in H$ then Cauchy-Schwarz gives

$$|\langle Tu, u\rangle| \le \|Tu\| \cdot \|u\|$$

and the "angle between" Tu and u" is

$$\cos^{-1}\left(\frac{\langle Tu, u\rangle}{\|Tu\| \cdot \|u\|}\right)$$

If $\dfrac{\langle Tu,u\rangle}{\|Tu\|\|u\|} = \pm 1$ then the angle between Tu and u is 0, ~~meaning~~ giving that

$Tu = \lambda u$.. Do you believe it?

An <u>orthonormal sequence in H</u> is a sequence $(a_1, a_2, \ldots)$ in H such that

$$\text{if } i,j \in \mathbb{Z}_{>0} \text{ then } \langle a_i, a_j \rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases}$$

<u>Theorem</u> Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H,

$$W = \text{span}\{a_1, a_2, \ldots\} \text{ and } \overline{W} = \overline{\text{span}\{a_1, a_2, \ldots\}}$$

Then $H = \overline{W} \oplus \overline{W}^\perp$ since

$$\text{if } x \in H \text{ then } x - \left(\sum_{n=1}^{\infty} \langle x, a_n \rangle a_n\right) \in \overline{W}^\perp.$$

<u>Gram-Schmidt</u>

Let H be a Hilbert space.

Let $(v_1, v_2, \ldots)$ be a sequence of linearly independent vectors in H. Define

$$a_1 = \frac{v_1}{\|v_1\|} \quad \text{and} \quad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \cdots - \langle v_{n+1}, a_n \rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1 \rangle a_1 - \cdots - \langle v_{n+1}, a_n \rangle a_n\|}$$

for $n \in \mathbb{Z}_{>0}$. Then $(a_1, a_2, a_3, \ldots)$ is an orthonormal sequence in H.