Let V be a normed vector space over $\mathbb{F}$.

The <u>dual of V</u> is

$$V^* = B(V, \mathbb{F}) = \{\varphi : V \to \mathbb{F} \mid \varphi \text{ is linear and } \|\varphi\| < \infty\}.$$

Let V and W be normed vector spaces over $\mathbb{F}$.

Let $T : V \to W$ be a bounded linear operator.

The <u>adjoint of T</u> is the function

$$T^* : W^* \to V^* \quad \text{given by} \quad (T^*\psi)(v) = (\psi \circ T)(v)$$

$$\begin{array}{ccc} V & \xrightarrow{T} & W \\ & & \downarrow \psi \\ & & \mathbb{F} \end{array}$$

<u>Finite dimensional vector spaces</u>

Let V and W be finite dimensional vector spaces,

$\{v_1, v_2, \ldots, v_n\}$ a basis of V

$\{w_1, w_2, \ldots, w_m\}$ a basis of W.

The <u>dual basis</u> to $\{v_1, v_2, \ldots, v_n\}$ is the basis $\{v^1, v^2, \ldots, v^n\}$ of V^* given by

$$v^i(v_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Let $\{w^1, \ldots, w^m\}$ be the dual basis to $\{w_1, \ldots, w_m\}$.

Let $T: V \to W$ be a linear operator and $T_{ij} \in \mathbb{F}$ given by

$$Tv_i = \sum_{j=1}^{m} T_{ji} w_j.$$

If $j, k \in \{1, \ldots, m\}$ then $(T^* w^j)(v_k) = w^j(Tv_k) = w^j\left(\sum_{l=1}^{m} T_{lk} w_l\right)$

$= T_{jk} = \sum_{i=1}^{m} T_{ji} v^i(v_k)$ giving that

$$T^* w^j = \sum_{i=1}^{n} T_{ji} v^i$$

Bilinear forms

A bilinear form $\langle\, , \rangle : V \times W \to \mathbb{F}$ is a function such that

(a) If $c_1, c_2 \in \mathbb{F}$, $v_1, v_2 \in V$ and $w \in W$ then

$$\langle c_1 v_1 + c_2 v_2, w\rangle = c_1 \langle v_1, w\rangle + c_2 \langle v_2, w\rangle.$$

(b) If $c_1, c_2 \in \mathbb{F}$ and $v \in V$ and $w_1, w_2 \in W$ then

$$\langle v, c_1 w_1 + c_2 w_2\rangle = c_1\langle v, w_1\rangle + c_2 \langle v, w_2\rangle.$$

Writing

$$\langle v, w\rangle = \psi_v(w)$$

gives a bijection

$$\left\{\begin{matrix}\text{bilinear forms on} \\ V \times W\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{linear transformations} \\ \text{from } V \text{ to } W^*\end{matrix}\right\}$$

$$\langle\, \rangle : V \times W \to \mathbb{F} \longleftrightarrow \begin{matrix} \psi_* : V \to W^* \\ v \mapsto \psi_v : W \to \mathbb{F} \\ \quad w \mapsto \langle v, w\rangle \end{matrix}$$

Let

$\langle\,,\rangle : V\times W \to \mathbb{F}$ be a bilinear form and

$\psi : V \to W^*$ the corresponding linear transformation.

The <u>orthogonal to W</u> with respect to $\langle\,,\rangle$ is

$$W^\perp = \ker \psi = \{v\in V \mid \text{if } w\in W \text{ then } \langle v,w\rangle = 0\}.$$

Hence $W^\perp = 0$ if and only if ψ is injective.

<u>Theorem</u> Let H be a Hilbert space and let $W\subseteq H$ be a subspace of H. If W is closed then

$$H = W\oplus W^\perp.$$

<u>Theorem</u> Let H be a Hilbert space. Then

$$H \xrightarrow{\psi} H^* \text{ is an isomorphism.}$$

An <u>orthonormal sequence in H</u> is a sequence $(a_1, a_2, \ldots)$ in H such that

$$\text{if } i,j\in\mathbb{Z}_{>0} \text{ then } \langle a_i, a_j\rangle = \delta_{ij} = \begin{cases} 0, & \text{if } i\neq j,\\ 1, & \text{if } i=j.\end{cases}$$

<u>Theorem</u> Let $(a_1, a_2, \ldots)$ be an orthonormal sequence in H,

$$W = \mathrm{span}\{a_1, a_2, \ldots\} \text{ and } \overline{W} = \overline{\mathrm{span}\{a_1, a_2, \ldots\}}.$$

Then $H = \overline{W}\oplus\overline{W}^\perp$ since

$$\text{if } x\in H \text{ then } x - \sum_{i=1}^{\infty}\langle x, a_i\rangle a_i \in \overline{W}^\perp.$$

Gram-Schmidt

Let H be a Hilbert space.
Let $(v_1, v_2, \dots)$ be a sequence of linearly independent vectors in H. Define

$$a_1 = \frac{v_1}{\|v_1\|} \quad \text{and} \quad a_{n+1} = \frac{v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n+1}, a_n\rangle a_n}{\|v_{n+1} - \langle v_{n+1}, a_1\rangle a_1 - \cdots - \langle v_{n+1}, a_n\rangle a_n\|}$$

for $n \in \mathbb{Z}_{>0}$. Then $(a_1, a_2, a_3, \dots)$ is an orthonormal sequence in H.