Metric and Hilbert Lecture 26, 23 September 2015

Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and let V and W be $\mathbb{K}$-vector spaces.

A linear operator from V to W is a function $T: V \to W$ such that

(a) if $v_1, v_2 \in V$ then $T(v_1+v_2) = T(v_1) + T(v_2)$,

(b) if $v \in V$ and $c \in \mathbb{K}$ then $T(cv) = cT(v)$.

Let $(V, \| \|_V)$ and $(W, \| \|_W)$ be normed $\mathbb{K}$-vector spaces.
The space of bounded operators from V to W is

$$B(V,W) = \{\text{linear operators } T: V \to W \mid \|T\| < \infty\}$$

where

$$\|T\| = \sup\left\{ \frac{\|Tv\|_W}{\|v\|_V} \mid v \in V \right\}$$

A Banach space is a normed vector space $(V, \| \|)$ which is complete (with metric given by $d(v_1, v_2) = \|v_2 - v_1\|$).
A Hilbert space is an inner product space $(V, \langle , \rangle)$ which is complete (with norm given by $\|v\| = \sqrt{\langle v, v\rangle}$).

Theorem Let V and W be normed vector spaces.
If W is a Banach space
then $B(V,W)$ is a Banach space.

<u>Theorem</u> Let V and W be normed vector spaces. and let $T: V \to W$ be a linear operator.

(a) $T \in B(V,W)$ if and only if T is continuous.

(b) T is continuous if and only if T is uniformly continuous.

<u>Proof</u> To show: (a) If $T \in B(V,W)$ then T is uniformly continuous.

(b) If T is uniformly continuous then T is continuous

(c) If T is continuous then $T \in B(V,W)$.

(b) For metric spaces, the definitions (epsilon-delta) of 'uniformly continuous' and 'continuous' can be set up in such a way that this follows directly from the definitions.

(a) Assume $T \in B(V,W)$.

To show: T is uniformly continuous.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x, y \in V$ and $d(x,y) < \delta$ then $d(Tx, Ty) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\delta \in \mathbb{R}_{>0}$ such that

if $x, y \in V$ and $d(x,y) < \delta$ then $d(Tx, Ty) < \varepsilon$.

Let $\delta = \frac{\varepsilon}{\|T\|}$.

To show: If $x, y \in V$ and $d(x,y) < \delta$ then $d(Tx, Ty) < \varepsilon$.

Assume $x, y \in V$ and $d(x,y) < \delta$.

To show: $d(Tx, Ty) < \varepsilon$.

$$d(Tx, Ty) = \|Tx - Ty\| = \|T(x-y)\|$$
$$\le \|T\| \, \|x-y\| = \|T\| \, d(x,y)$$
$$< \|T\| \cdot \delta < \varepsilon$$

So T is uniformly continuous.

(c) To show: If T is continuous then $T \in B(V, W)$.

Assume T is continuous.

To show: $\|T\| < \infty$.

To show: There exists $C \in \mathbb{R}_{>0}$ such that

if $u \in V$ then $\|Tu\| \le C\|u\|$

Since T is continuous, T is continuous at 0.

So there exists $\delta \in \mathbb{R}_{>0}$ such that

if $x \in V$ and $\|x\| < \delta$ then $\|Tx\| < 1$.

Let $c = \frac{2}{\delta}$

To show: If $u \in V$ then $\|Tu\| \le c\|u\|$.

Assume $u \in V$.

To show: $\|Tu\| \le c\|u\|$.

Let $x = \frac{\delta}{2} \frac{u}{\|u\|}$ so that $\|x\| < \frac{\delta}{2}$

Then $1 > \|Tx\| = \|T(\frac{\delta}{2} \frac{u}{\|u\|})\| = \frac{\delta}{2\|u\|} \|Tu\|$

So $\|Tu\| < \frac{2}{\delta} \|u\| = c\|u\|$.

So T is bounded //.