Metric and Hilbert, Lecture 25, 22 September 2015 (1)

Examples Univ. of Melbourne.

Normed vector spaces: $\mathbb{R}^n$ with

$$\|x\| = \left(|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2\right)^{\frac{1}{2}} \quad \text{if } x = (x_1, x_2, \ldots, x_n).$$

Positive definite symmetric inner product spaces

(a) $V = \mathbb{R}^n$ is an $\mathbb{R}$-vector space with

$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$, if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

(b) $V = \mathbb{C}^n$ is a $\mathbb{C}$-vector space with

$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n$, if $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$.

Positive definite Hermitian inner product spaces

(a) $V = \mathbb{R}^n$ as an $\mathbb{R}$-vector space with

$\langle x, y\rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$, if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

(b) $V = \mathbb{C}^n$ as a $\mathbb{C}$-vector space with

$\langle x, y\rangle = x_1 \bar{y}_1 + \cdots + x_n \bar{y}_n$, if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$

(Note: $\bar{y}_i = y_i$ if $y_i \in \mathbb{R}$, as in case (a)).

<u>Bases</u>: Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$.

Let V be a $\mathbb{K}$-vector space.

A (Hamel) <u>basis</u> of V is a subset $B \subseteq V$ such that

(a) $\mathbb{K}\text{span}(B) = V$

(b) B is linearly independent,

where

$$\mathbb{K}\text{-span}(B) = \left\{ a_1 b_1 + \cdots + a_\ell b_\ell \;\middle|\; \begin{array}{l} \ell \in \mathbb{Z}_{>0},\ b_1, \ldots, b_\ell \in B, \\ a_1, \ldots, a_\ell \in \mathbb{K} \end{array} \right\}$$

and B is <u>linearly independent</u> if B satisfies

if $\ell \in \mathbb{Z}_{>0}$ and $b_1, \ldots, b_\ell \in B$ and $a_1, \ldots, a_\ell \in \mathbb{K}$ and $a_1 b_1 + \cdots + a_\ell b_\ell = 0$ then $a_1 = 0, a_2 = 0, \ldots, a_\ell = 0$.

A <u>Schauder basis</u> of V is a sequence $(b_1, b_2, \ldots)$ in V such that

if $v \in V$ then there exists a unique sequence $(a_1, a_2, \ldots)$ in $\mathbb{K}$ such that $\sum_{i=1}^{\infty} a_i b_i = v$.

<u>Note</u>: $v = \sum_{i=1}^{\infty} a_i b_i$ means $v = \lim_{n \to \infty} s_n$ where

$s_1 = a_1 b_1$, $s_2 = a_1 b_1 + a_2 b_2$, ... Hence,

if $(b_1, b_2, \ldots)$ is a Schauder basis then $V \subseteq \overline{\mathbb{K}\text{span}\{b_1, b_2, \ldots\}}$

When is $\overline{\mathbb{K}\text{-span}(B)} = V$?

Let $e_1 = (1,0,0,\ldots)$, $e_2 = (0,1,0,0,\ldots)$, ...

with e_i having 1 in the i^{th} entry and all other entries 0.

Then

$$c_c = \mathbb{R}\text{-span}\{e_1, e_2, \ldots\} = \left\{ \begin{array}{l} \text{sequences } (a_1, a_2, \ldots) \in \mathbb{R}^\infty \text{ with} \\ \text{all but a finite number of entries} \\ \text{equal to } 0. \end{array} \right\}$$

Then,

in ℓ^1, $\overline{\mathbb{R}\text{-span}\{e_1, e_2, \ldots\}} = \ell^1$,

in ℓ^p with $p \in \mathbb{R}_{>0}$, then $\overline{\mathbb{R}\text{-span}\{e_1, e_2, \ldots\}} = \ell^p$,

in ℓ^∞, then $\overline{\mathbb{R}\text{-span}\{e_1, e_2, \ldots\}} = c_0$, where

$$c_0 = \{ (a_1, a_2, \ldots) \in \mathbb{R}^\infty \mid \lim_{n\to\infty} a_n = 0 \}.$$

is the set of sequences in $\mathbb{R}$ which converge to 0.

Since $(1,1,1,\ldots) \in \ell^\infty$ but $(1,1,1,\ldots) \notin c_0$ then $c_0 \subsetneq \ell^\infty$.

So $\overline{\mathbb{R}\text{-span}\{e_1, e_2, \ldots\}} \neq \ell^\infty$.

<u>Proposition</u> Let $(V, \|\ \|)$ be a normed vector space.

Then

V has a countable dense set C

if and only if

V has a countable subset B with $\overline{\mathbb{K}\text{-span}(B)} = V$.

<u>Proof</u> $\Rightarrow$ Assume C is a countable dense subset of V.

To show: There is a countable subset $B \subseteq V$ with

$\overline{\mathbb{K}\text{-span}(B)} = V$.

Let $B = C$.

To show: $\overline{\mathbb{K}\text{-span}(B)} = V$.

Since $C \subseteq \mathbb{K}\text{span}(C) = \mathbb{K}\text{-span}(B)$ then

$V = \overline{C} \subseteq \overline{\mathbb{K}\text{-span}(B)}$. So $V = \overline{\mathbb{K}\text{-span}(B)}$.

$\Leftarrow$ Assume V has a countable subset B with $\overline{\mathbb{K}\text{-span}(B)} = V$.

To show: V has a countable dense set C.

Let $\mathbb{F} = \mathbb{Q}$ if $\mathbb{K} = \mathbb{R}$ and let $\mathbb{F} = \mathbb{Q} + i\mathbb{Q}$ if $\mathbb{K} = \mathbb{C}$.

Let $C = \mathbb{F}\text{-span}(B)$.

Then C is countable and

$\overline{C} = \overline{\mathbb{F}\text{-span}(B)} \supseteq \overline{\mathbb{F}\text{-span}(B)} = \overline{\mathbb{K}\text{-span}(B)} = V$.

So $\overline{C} = V$. So C is dense in V. //