Metric and Hilbert Spaces, Lecture 24, 17 September 2015
Univ. of Melbourne

Theorem Let $(V, \|\ \|)$ be a normed vector space and let $d: V\times V \to \mathbb{R}_{\ge 0}$ be the metric on V given by

$$d(x,y) = \|y-x\|.$$

Then V is a complete metric space if and only if V satisfies

If $(a_1, a_2, \ldots)$ is a sequence in V and

(*) $\sum_{i\in\mathbb{Z}_{>0}} \|a_i\|$ converges then $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

Proof $\Rightarrow$ Assume V is complete

To show: V satisfies (*).

Assume $(a_1, a_2, \ldots)$ is a sequence in V and

$\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

To show: $\sum_{i\in\mathbb{Z}_{>0}} a_i$ converges.

Let

$$s_n = \sum_{i=1}^{n} a_i \quad \text{and} \quad S_n = \sum_{i=1}^{n} \|a_n\|.$$

Since the sequence $(S_1, S_2, \ldots)$ converges,

the sequence $(S_1, S_2, \ldots)$ is Cauchy.

Since

$$\|s_n - s_m\| = \Big\| \sum_{i=m+1}^{n} a_i \Big\| \leq \sum_{i=m+1}^{n} \|a_i\| = \| S_n - S_m \|$$

then the sequence $(s_1, s_2, \ldots)$ is Cauchy.

Since V is complete, the sequence $(s_1, s_2, \ldots)$ converges.

So $\sum_{i \in \mathbb{Z}_{>0}} a_i$ converges.

$\Leftarrow$ Assume that V satisfies $(*)$.

To show: V is complete.

Let $(s_1, s_2, \ldots)$ be a Cauchy sequence in V.

To show: $(s_1, s_2, \ldots)$ converges.

Using that $(s_1, s_2, \ldots)$ is Cauchy, let

$k_n \in \mathbb{Z}_{>0}$ be such that

if $r, m \in \mathbb{Z}_{\geq k_n}$ then $\|s_r - s_m\| < \frac{1}{2^n}$

Let

$a_1 = s_{k_1}$, $a_2 = s_{k_2} - s_{k_1}$, $a_3 = s_{k_3} - s_{k_2}$, ...

Then $\|a_n\| < \frac{1}{2^n}$.

So $$\sum_{n \in \mathbb{Z}_{>0}} \|a_n\| < \sum_{n \in \mathbb{Z}_{>0}} \frac{1}{2^n} = 1.$$

So $\sum_{n \in \mathbb{Z}_{>0}} \|a_n\|$ converges.

Since V satisfies $(*)$ then $\sum_{n \in \mathbb{Z}_{>0}} a_n$ converges

So the sequence $(s_{k_1}, s_{k_2}, \ldots)$ converges, since

$$s_{k_1} = a_1, \quad s_{k_2} = a_1 + a_2, \quad s_3 = a_1 + a_2 + a_3, \ldots$$

So the sequence $(s_1, s_2, s_3, \ldots)$ has a cluster point.

Since $(s_1, s_2, \ldots)$ is Cauchy and has a cluster point then $(s_1, s_2, \ldots)$ converges.

So V is complete. //