<u>Theorem</u> Let $X = \mathbb{R}$ with metric given by $d(x,y) = |y-x|$.

(a) $\mathbb{R}$ is a complete metric space.

(b) $E \subseteq \mathbb{R}$ is connected if and only if E is an interval.

(c) $E \subseteq \mathbb{R}$ is compact if and only if E is closed and bounded.

<u>Proof of (b)</u>

$\Rightarrow$ Assume E is not an interval.

Let $x, y \in E$ and $z \in \mathbb{R}$ with

$$x < z < y, \quad \text{and} \quad z \notin E.$$

Let $A = (-\infty, z) \cap E$ and $B = (z, \infty) \cap E$

Then A and B are open subsets of E and

$$A \neq \emptyset, \quad B \neq \emptyset, \quad A \cap B = \emptyset \quad \text{and} \quad A \cup B = E.$$

So E is not connected.

$\Leftarrow$ Assume E is an interval.

To show: E is connected.

Proof by contradiction

Assume E is not connected.

Let $A \subseteq E$ and $B \subseteq E$ be open subsets of E such that $A \neq \phi$, $B \neq \phi$, $A \cap B = \phi$ and $A \cup B = E$.

Then $f: E \to \{0, 1\}$ given by

$$f(z) = \begin{cases} 0, & \text{if } z \in A \\ 1, & \text{if } z \in B \end{cases}$$ is a continuous surjective function

Let $x_1, y_1 \in J$ with $f(x_1) = 0$ and $f(y_1) = 1$

i.e. $x_1 \in A$ and $y_1 \in B$

Switching A and B if necessary we may assume that $x_1 < y_1$

Construct sequences $x_1, x_2, \dots$ and $y_1, y_2, \dots$ by

$x_{i+1} = \frac{x_i + y_i}{2}$, $y_{i+1} = y_i$, if $f\left(\frac{x_i + y_i}{2}\right) = 0$ i.e. $\frac{x_i+y_i}{2} \in A$

$x_{i+1} = x_i$; $y_{i+1} = \frac{x_i + y_i}{2}$, if $f\left(\frac{x_i + y_2}{2}\right) = 1$ i.e. $\frac{x_i+y_i}{2} \in B$

By induction, $x_i \in E$ and $y_i \in E$ and

since E is an interval $\frac{1}{2}(x_i + y_i) \in E$.

So $f\left(\frac{x_i + y_i}{2}\right)$ is defined and $x_{i+1} \in E$ and $y_{i+1} \in E$.

$f(x_{i+1}) = 0$, $f(y_{i+1}) = 1$ and $x_i \leq x_{i+1} < y_{i+1} \leq y_i$

and

$|x_{i+1} - y_{i+1}| \leq \frac{1}{2}|x_i - y_i|$ so that $|x_{i+1} - y_{i+1}| \leq \frac{1}{2^i}|x_1 - y_1|$.

Then

$x_1, x_2, \ldots$ is increasing and bounded by y_1, so

$\lim_{n\to\infty} x_n$ exists in $\mathbb{R}$ (since $\mathbb{R}$ is complete).

$y_1, y_2, \ldots$ is decreasing and bounded by x_1, so

$\lim_{n\to\infty} y_n$ exists in $\mathbb{R}$ (since $\mathbb{R}$ is complete).

Since

$$\lim_{n\to\infty} |x_n - y_n| = 0 \quad \text{then} \quad \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n.$$

Let

$$z = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

Since $x_1 \le x_2 \le \cdots \le x_n < y_n \le y_{n-1} \le \cdots \le y_1$ for $n \in \mathbb{Z}_{>0}$ then

$$x_1 < z < y_1$$

Since E is an interval then $z \in E$.

Since f is continuous

$$0 = \lim_{n\to\infty} f(x_n) = f(z) = \lim_{n\to\infty} f(y_n) = 1.$$

This is a contradiction. i.e. $z \in \bar{A} = A$ and

$z \in \bar{B} = B$

so that $z \in A \cap B$.

So E is connected. //

Theorem Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f: X \to Y$ be a continuous function. Let $E \subseteq X$.

(a) If E is connected then $f(E)$ is connected

(b) If E is compact then $f(E)$ is compact.

Theorem Let $A \subseteq \mathbb{R}$. Then A is connected and compact if and only if

A is a closed and bounded interval.

i.e. A is connected and compact if and only if there exist $m, M \in \mathbb{R}$ such that $A = [m, M]$.

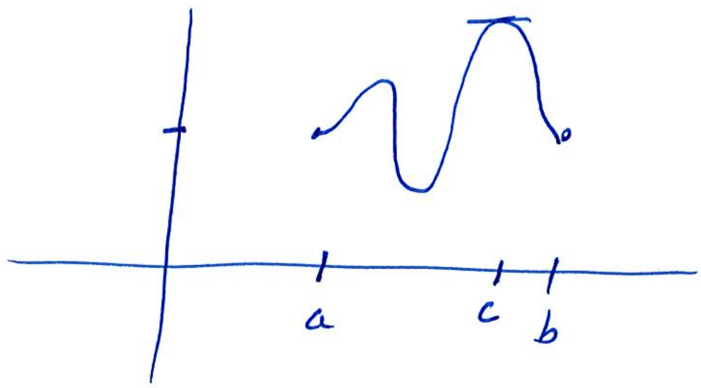
Theorem (Intermediate value theorem).
Let $a, b \in \mathbb{R}$ with $a < b$.

(a) Show that if $f: [a,b] \to \mathbb{R}$ is a continuous function and $w \in (f(a), f(b))$ then there exists $c \in (a,b)$ such that $f(c) = w$.

(b) If $f: [a,b] \to \mathbb{R}$ is a continuous function then there exist $m, M \in \mathbb{R}$ such that

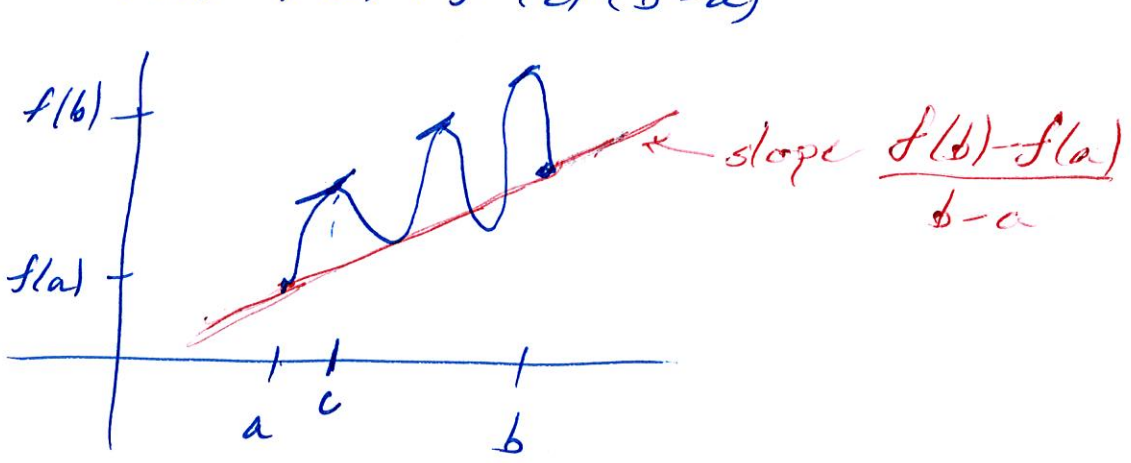
$$f([a,b]) = [m, M].$$

<u>Theorem</u> (Rolle's theorem) Let $a,b \in \mathbb{R}$ with $a<b$.
If $f:[a,b] \to \mathbb{R}$ is a function such that
f is continuous and $f':(a,b) \to \mathbb{R}$ exists and
$f(a)=f(b)$ then there exists $c \in (a,b)$
such that $f'(c)=0$.



<u>Theorem</u> (Mean value theorem) Let $a,b \in \mathbb{R}$ with $a<b$.
If $f:[a,b] \to \mathbb{R}$ is a function such that
f is continuous and $f':(a,b) \to \mathbb{R}$ exists
then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(c)(b-a)$$



__Theorem__ (Taylor's theorem) Let $a, b \in \mathbb{R}$ with $a < b$. Let $N \in \mathbb{Z}_{\geq 0}$. If $f : [a,b] \to \mathbb{R}$ is a function such that

$f^{(N)} : [a,b] \to \mathbb{R}$ is continuous and

$f^{(N+1)} : (a,b) \to \mathbb{R}$ exists

then there exists $c \in (a,b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2!} f''(a)(b-a)^2 + \dots$$

$$+ \frac{1}{N!} f^{(N)}(a)(b-a)^N + \frac{1}{(N+1)!} f^{(N+1)}(c)(b-a)^{N+1}.$$