Metric and Hilbert spaces, Lecture 2, 10 September 2015 (1)

Theorem $\mathbb{R}$, with the standard metric, is complete.

Proof

To show: If $\vec{x}_1, \vec{x}_2, \ldots$ is a Cauchy sequence in $\mathbb{R}$ then $\vec{x}_1, \vec{x}_2, \ldots$ converges

Assume

$$\begin{aligned} \vec{x}_1 &= (x_{11}, x_{12}, \ldots) \\ \vec{x}_2 &= (x_{21}, x_{22}, \ldots) \\ \vec{x}_3 &= (x_{31}, x_{32}, \ldots) \\ &\vdots \end{aligned}$$

is a Cauchy sequence in $\mathbb{R}$.

To show: There exists $\vec{z} = (z_1, z_2, \ldots)$ in $\mathbb{R}$ such that $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

Let $z_k \in \mathbb{Q}_{sm}$ such that $d(z_k, \vec{x}_k) < \frac{1}{10^k}$

To show: (a) $\vec{z} = (z_1, z_2, \ldots)$ is a Cauchy sequence

(b) $\lim_{n\to\infty} \vec{x}_n = \vec{z}$.

(a) To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>\ell}$ then $d(z_r, z_s) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{>\ell}$ then $d(z_r, z_s) < \varepsilon$.

Let $N_1 \in \mathbb{Z}_{>0}$ so that $\frac{1}{10^{N_1}} < \frac{\varepsilon}{3}$

Let $N_2 \in \mathbb{Z}_{>0}$ be such that if $r, s \in \mathbb{Z}_{\geq N_2}$

then $\hat{d}(\vec{x}_r, \vec{x}_s) < \frac{\varepsilon}{3}$.

Let $\ell = \mathrm{Max}(N_1, N_2)$.

To show: If $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(z_r, z_s) < \varepsilon$.

Assume $r, s \in \mathbb{Z}_{\geq \ell}$

To show: $d(z_r, z_s) < \varepsilon$.

$$d(z_r, z_s) = \hat{d}(\varphi(z_r), \varphi(z_s)) \cancel{\leq \hat{d}(\varphi(z_r), \vec{x}_r) + d}$$

$$\leq \hat{d}(\varphi(z_r), \vec{x}_r) + d(\vec{x}_r, \vec{x}_s) + \hat{d}(\vec{x}_s, \varphi(z_s))$$

$$\leq \frac{1}{10^r} + \frac{\varepsilon}{3} + \frac{1}{10^s} < \frac{1}{10^{\ell_1}} + \frac{\epsilon}{3} + \frac{1}{10^{\ell_2}} < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

So $\vec{z}$ is a Cauchy sequence.

(b) To show: $\lim_{n \to \infty} \vec{x}_n = \vec{z}$.

To show: $\lim_{n \to \infty} \hat{d}(\vec{x}_n, \vec{z}) = 0$.

$$\lim_{n \to \infty} \hat{d}(\vec{x}_n, \vec{z}) \leq \lim_{n \to \infty} \left(\hat{d}(\hat{x}_n, \varphi(z_n)) + \hat{d}(\varphi(z_n), \vec{z})\right)$$

$$\leq \lim_{n \to \infty} \left(\frac{1}{10^n} + \hat{d}(\varphi(z_n), \vec{z})\right)$$

$$= \lim_{n \to \infty} \frac{1}{10^n} + \lim_{n \to \infty} \hat{d}(\varphi(z_n), \vec{z}) = 0 + 0 + 0.$$

So $\mathbb{R}$ with $\hat{d}$ is complete.