Completion of a metric space

Let (X, d_X) and (Y, d_Y) be metric spaces.

An <u>isometry from X to Y</u> is a function $\varphi : X \to Y$ such that

if $x_1, x_2 \in X$ then $d_Y(\varphi(x_1), \varphi(x_2)) = d_X(x_1, x_2)$.

Let (X,d) be a metric space. The <u>completion of (X,d)</u> is a metric space with an isometry

$$\iota : X \to \hat{X} \text{ such that } (\hat{X}, \hat{d}) \text{ is complete and}$$

$$\overline{\iota(X)} = \hat{X}.$$

Existence of the completion of (X,d)

The <u>completion</u> <u>of (X,d)</u> is the metric space

$\hat{X} = \{\text{Cauchy sequences in } X\}$ with

$$\begin{aligned} \iota : X &\longrightarrow \hat{X} \\ x &\longmapsto (x, x, x, \ldots) \end{aligned}$$

where $\hat{X}$ has the metric $\hat{d} : \hat{X} \times \hat{X} \to \mathbb{R}_{\geq 0}$ given by

$$\hat{d}(\vec{x}, \vec{y}) = \lim_{n \to \infty} d(x_n, y_n)$$

and Cauchy sequences $\vec{x} = (x_1, x_2, \ldots)$ and $\vec{y} = (y_1, y_2, \ldots)$ are <u>equal</u> <u>in $\hat{X}$</u>,

$\vec{x} = \vec{y}$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Uniqueness of the completion of (X,d)

Let (X,d) be a metric space.

Let $(\hat{X}_1, \hat{d}_1, \hat{\iota}_1)$ and $(\hat{X}_2, \hat{d}_2, \hat{\iota}_2)$ be completions of (X,d). Then there exists a bijective isometry

$$\hat{X}_1 \xrightarrow{\varphi} \hat{X}_2.$$

<u>HW</u>: If $\varphi: X \to Y$ is an isometry then φ is injective

<u>Proof</u> Assume $\varphi: X \to Y$ is an isometry.

To show: φ is injective

To show: If $x_1, x_2 \in X$ and $\varphi(x_1) = \varphi(x_2)$ then $x_1 = x_2$

Assume $x_1, x_2 \in X$ and $\varphi(x_1) = \varphi(x_2)$

To show: $x_1 = x_2$

To show: $d_X(x_1, x_2) = 0$.

$$d_X(x_1, x_2) = d_Y(\varphi(x_1), \varphi(x_2)) = 0.$$

So $x_1 = x_2$.

So φ is injective //

To show: There exists $\varphi : \hat{X}_1 \to \hat{X}_2$ such that φ is a bijective isometry.

The key point in the definition of φ:

Since $\overline{\imath_1(X)} = \hat{X}_1$ then,

if $z \in \hat{X}_1$ then there exists $\imath_1(x_1), \imath_1(x_2), \ldots$ such that $\lim_{n\to\infty} \imath_1(x_n) = z$.

Then define

$$\varphi(z) = \varphi\left(\lim_{n\to\infty} \imath_1(x_n)\right) = \lim_{n\to\infty} \varphi(\imath_1(x_n))$$

$$= \lim_{n\to\infty} \imath_2 \imath_1^{-1} \imath_1(x_n) = \lim_{n\to\infty} \imath_2(x_n).$$

i.e.

$$\varphi\left(\lim_{n\to\infty} \imath_1(x_n)\right) = \lim_{n\to\infty} \imath_2(x_n).$$

Similarly,

$$\hat{d}_1\left(\lim_{k\to\infty} \imath_1(x_{1k}), \lim_{\ell\to\infty} \imath_1(x_{2\ell})\right)$$

$$= \lim_{k\to\infty}\left(\lim_{\ell\to\infty} \hat{d}_1(\imath_1(x_{1k}), \imath_1(x_{2\ell}))\right) = \lim_{k\to\infty}\left(\lim_{\ell\to\infty} d(x_{1k}, x_{2\ell})\right)$$

$$= \lim_{k\to\infty}\left(\lim_{\ell\to\infty} \hat{d}_2(\imath_2(x_{1k}), \imath_2(x_{2\ell}))\right) = \hat{d}_2\left(\lim_{k\to\infty} \imath_2(x_{1k}), \lim_{\ell\to\infty} \imath_2(x_{2\ell})\right).$$

To show: $\varphi: \hat{X}_1 \to \hat{X}_2$ is surjective.

To show: If $z \in \hat{X}_2$ then there exists $w \in \hat{X}_1$ such that $\varphi(w) = z$.

Assume $z \in \hat{X}_2$.

To show: There exists $w \in \hat{X}_1$ such that $\varphi(w) = z$.

Since $\hat{X}_2 = \overline{\iota_2(X)} = \{y \in X_2 \mid$ there exists $\iota_2(x_1), \iota_2(x_2), \ldots$ in $\iota_2(X)$ with $\lim_{n\to\infty} \iota_2(x_n) = y\}$

then there exists $(\iota_2(x_1), \iota_2(x_2), \ldots)$ in $\iota_2(X)$ such that $\lim_{n\to\infty} \iota_2(x_n) = z$.

Let $w = \lim_{n\to\infty} \iota_1(x_n)$ in $\hat{X}_1$.

To show: $\varphi(w) = z$.

$$\varphi(w) = \varphi\left(\lim_{n\to\infty} \iota_1(x_n)\right) = \lim_{n\to\infty} \varphi(\iota_1(x_n))$$

$$= \lim_{n\to\infty} \iota_2(\iota_1^{-1}(\iota_1(x_n))) = \lim_{n\to\infty} \iota_2(x_n) = z.$$

So φ is surjective.

To show: $\varphi : \hat{X}_1 \to \hat{X}_2$ is an isometry.

To show: If $w_1, w_2 \in \hat{X}_1$ then $\hat{d}_2(\varphi(w_1), \varphi(w_2)) = \hat{d}_1(w_1, w_2)$.

Assume $w_1, w_2 \in \hat{X}_1$.

Using that $\hat{X}_1 = \overline{z_1(X)}$ let

$(z_1(x_{11}), z_1(x_{12}), \ldots)$ in $z_1(X)$ such that $\lim_{n\to\infty} z_1(x_{1n}) = w_1$,

$(z_1(x_{21}), z_1(x_{22}), \ldots)$ in $z_1(X)$ such that $\lim_{n\to\infty} z_1(x_{2n}) = w_2$.

To show: $\hat{d}_2(\varphi(w_1), \varphi(w_2)) = \hat{d}_1(w_1, w_2)$.

$$\hat{d}_2(\varphi(w_1), \varphi(w_2)) = \hat{d}_2\left(\varphi\left(\lim_{k\to\infty} z_1(x_{1k})\right), \varphi\left(\lim_{l\to\infty} z_1(x_{2l})\right)\right)$$

$$= \hat{d}_2\left(\lim_{k\to\infty} \varphi(z_1(x_{1k})), \lim_{l\to\infty} \varphi(z_1(x_{2l}))\right)$$

$$= \hat{d}_2\left(\lim_{k\to\infty} z_2(x_{1k}), \lim_{l\to\infty} z_2(x_{2l})\right)$$

$$= \lim_{k\to\infty}\left(\hat{d}_2\left(z_2(x_{1k}), \lim_{l\to\infty} z_2(x_{2l})\right)\right)$$

$$= \lim_{k\to\infty}\left(\lim_{l\to\infty} \hat{d}_2(z_2(x_{1k}), z_2(x_{2l}))\right)$$

$$= \lim_{k\to\infty}\left(\lim_{l\to\infty} d(x_{1k}, x_{2l})\right)$$

$$= \lim_{k\to\infty}\left(\lim_{l\to\infty} \hat{d}_1(z_1(x_{1k}), z_1(x_{2l}))\right)$$

$$= \lim_{k\to\infty}\left(\cancel{\lim}\ \hat{d}_1\left(z_1(x_{1k}), \lim_{l\to\infty} z_1(x_{2l})\right)\right).$$

$$= \hat{d}_1\left(\lim_{k\to\infty} z_1(x_{1k}), \lim_{l\to\infty} z_1(x_{2l})\right)$$

$$= \hat{d}_1(w_1, w_2).$$

So φ is an isometry.