Metric and Hilbert spaces, Lecture 19, 08.09.2015 ①
Sequences of functions Univ. of Melbourne

Examples (1) $f_1, f_2, \ldots$ defined by $f_n : [0,1) \to [0,1)$, $x \mapsto x^n$

(2) $f_1, f_2, \ldots$ defined by $f_n : [0,1] \to [0,1]$, $x \mapsto x^n$

(3) $f_1, f_2, \ldots$ defined by $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, $x \mapsto x^n$

Let (X, d_X) and (Y, d_Y) be metric spaces.
Let
$$F = \{\text{functions } f : X \to Y\}$$
and define $d_\infty : F \times F \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ by
$$d_\infty(f,g) = \sup\{d_Y(f(x), g(x)) \mid x \in X\}.$$

Let $f_1, f_2, \ldots$ be a sequence in F. Let $f \in F$.
The sequence $f_1, f_2, \ldots$ converges pointwise to f if $f_1, f_2, \ldots$ satisfies:
if $x \in X$ then $\lim_{n\to\infty} d_Y(f_n(x), f(x)) = 0$

The sequence $f_1, f_2, \ldots$ converges uniformly to f if $f_1, f_2, \ldots$ satisfies:
$$\lim_{n\to\infty} d_\infty(f_n, f) = 0.$$

Look up:

www.ms.unimelb.edu.au/~ram/Teaching/2014MetricandHilbert/Ass1SOLNSnos5to10.pdf.

<u>HW</u>: Show that $(f_1, f_2, \ldots)$ converges pointwise to f if and only if $(f_1, f_2, \ldots)$ satisfies:

if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq N}$ then $d_Y(f_n(x), f(x)) < \varepsilon$.

<u>HW</u>: Show that $(f_1, f_2, \ldots)$ converges uniformly to f if and only if $(f_1, f_2, \ldots)$ satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $x \in X$

and $n \in \mathbb{Z}_{>0}$ and $n \in \mathbb{Z}_{\geq N}$ then $d_Y(f_n(x), f(x)) < \varepsilon$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A <u>continuous function from X to Y</u> is
a function $f : X \to Y$ such that

if $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.

Let $(X, \mathcal{X}_X)$ and $(Y, \mathcal{X}_Y)$ be uniform spaces.
A <u>uniformly continuous function from X to Y</u> is
a function $f : X \to Y$ such that

if $V \in \mathcal{X}_Y$ then $(f \times f)^{-1}(V) \in \mathcal{X}_X$.

Let (X, d_X) be a metric space.
The <u>metric space uniformity on X</u> is

$$\mathcal{X}_X = \{ V \subseteq X \times X \mid V \text{ contains an } \varepsilon\text{-diagonal} \}$$

where an <u>ε-diagonal</u> is

$$B_\varepsilon = \{ (x_1, x_2) \in X \times X \mid d(x_1, x_2) < \varepsilon \}$$

The <u>metric space topology on X</u> is

$$\mathcal{T}_X = \{ \text{unions of open balls} \}$$

where an <u>open ball</u> is

$$B_\varepsilon(x) = \{ y \in X \mid d(x, y) < \varepsilon \}.$$

<u>HW</u>: Let (X, d_X) and (Y, d_Y) be metric spaces.
Let $f: X \to Y$ be a function. Show that
$f: X \to Y$ is continuous if and only if f satisfies:

if $x \in X$ and $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$
such that
if $y \in X$ and $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

<u>HW</u>: Let (X, d_X) and (Y, d_Y) be metric spaces.
Let $f: X \to Y$ be a function. Show that
$f: X \to Y$ is uniformly continuous if and only if
f satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$
such that
if $x \in X$
and $y \in X$ and $d_X(x, y) < \delta$ then $d_Y(f(x), f(y)) < \varepsilon$.

<u>Examples</u> (a) $\begin{matrix} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^2 \end{matrix}$ and $\begin{matrix} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x, y) \mapsto xy \end{matrix}$
are continuous but not uniformly continuous.

(b) $\begin{matrix} \mathbb{R} \to \mathbb{R} \\ x \mapsto -x \end{matrix}$ and $\begin{matrix} \mathbb{R} \times \mathbb{R} \to \mathbb{R} \\ (x, y) \mapsto x+y \end{matrix}$
are continuous and also uniformly continuous.