Metric and Hilbert, Lecture 17, 2 Sept 2015

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Theorem Let (X,d) be a complete metric space. Then

X is compact if and only if X is ball compact.

Proof ⇒: Assume X is cover compact.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exist $x_1, x_2, \ldots, x_\ell \in X$ such that $X \subseteq B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell)$.

Assume $\varepsilon \in \mathbb{R}_{>0}$. Then

$$\mathcal{S} = \{ B_\varepsilon(x) \mid x \in X \} \text{ is an open cover.}$$

So $\mathcal{S}$ contains a finite subcover of X.

So there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in X$ such that

$$X \subseteq B_\varepsilon(x_1) \cup \cdots \cup B_\varepsilon(x_\ell).$$

So X is ball compact.

⇐ Assume X is ball compact.

To show: X is sequentially compact.

To show: If $(x_1, x_2, \ldots)$ is a sequence in X then $(x_1, x_2, \ldots)$ has a cluster point in X.

Assume $(x_1, x_2, \ldots)$ is a sequence in X.

To show: $(x_1, x_2, \ldots)$ has cluster point in X.

To show: There exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ such that $(x_{n_1}, x_{n_2}, \ldots)$ converges in X.

Using that X is ball compact,

Let $n_1 \in \mathbb{Z}_{>0}$ be minimal such that $B_1(x_{n_1})$ contains an infinite number of $(x_1, x_2, \ldots)$

Let $n_2 \in \mathbb{Z}_{>0}$ be minimal such that $n_2 > n_1$ and $B_{\frac{1}{2}}(x_{n_2})$ contains an infinite number of $(x_1, x_2, \ldots) \cap B_1(x_{n_1})$.

Let $n_3 \in \mathbb{Z}_{>0}$ be minimal such that $n_3 > n_2$ and $B_{\frac{1}{3}}(x_{n_3})$ contains an infinite number of $(x_1, x_2, \ldots) \cap B_1(x_{n_1}) \cap B_{\frac{1}{2}}(x_{n_2})$.

To show: $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ converges.

To show: $(x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$, since (X, d) is complete.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_{n_r}, x_{n_s}) < \varepsilon$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\ell \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_{n_r}, x_{n_s}) < \varepsilon$.

Let $\ell \in \mathbb{Z}_{>0}$ be such that $\frac{1}{\ell} < \frac{\varepsilon}{2}$.

To show: If $r, s \in \mathbb{Z}_{\geq \ell}$ then $d(x_{n_r}, x_{n_s}) < \varepsilon$.

Assume $r, s \in \mathbb{Z}_{\geq \ell}$.

To show: $d(x_{n_r}, x_{n_s}) < \varepsilon$.

$$d(x_{n_r}, x_{n_s}) \leq d(x_{n_r}, x_{n_\ell}) + d(x_{n_\ell}, x_{n_s}) < \frac{1}{\ell} + \frac{1}{\ell}$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $(x_{n_1}, x_{n_2}, \dots)$ is Cauchy.

So $(x_{n_1}, x_{n_2}, \dots)$ converges.

So $(x_1, x_2, \dots)$ has a cluster point.

So (X,d) is sequentially compact. //

<u>Proposition</u> Let (X,d) be a metric space.

If X is sequentially compact then X is ball compact.

<u>Proof</u> Assume X is not ball compact.

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that X is not covered by finitely many $B_\varepsilon(x)$.

Let $x_1 \in X$, $x_2 \in B_{\frac{\varepsilon}{2}}(x_1)^c$, $x_3 \in \left(B_{\frac{\varepsilon}{2}}(x_1) \cup B_{\frac{\varepsilon}{2}}(x_2)\right)^c, \dots$

Then $(x_1, x_2, \dots)$ has no cluster point, since every $B_{\frac{\varepsilon}{2}}(x)$ contains at most one point of $(x_1, x_2, \dots)$.

So X is not sequentially compact.