Let (X,d) be a metric space and let $(x_1, x_2, \ldots)$ be a sequence in X. Let $y \in X$.

A cluster point of $(x_1, x_2, \ldots)$ is $y \in X$ such that

there exists a subsequence $(x_{n_1}, x_{n_2}, \ldots)$ of $(x_1, x_2, \ldots)$ such that $y = \lim_{k\to\infty} x_{n_k}$.

Example In $\mathbb{R}$ let

$$(x_1, x_2, \ldots) = (2, \tfrac{-3}{2}, 1+\tfrac{1}{3}, -(1+\tfrac{1}{4}), 1+\tfrac{1}{5}, -(1+\tfrac{1}{6}), \ldots)$$

Then 1 and -1 are cluster points of $(x_1, x_2, \ldots)$ but $(x_1, x_2, \ldots)$ has no limit.

Let $A \subseteq X$. Let $\mathcal{T}$ be the metric space topology on X.

(a) The set A is sequentially compact if every sequence in A has a cluster point in A.

(b) The set A is Cauchy compact, or complete, if every Cauchy sequence has a cluster point in A

(c) The set A is ball compact, or precompact, or totally bounded, if A satisfies if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots, x_\ell \in X$ such that

$$A \subseteq B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_\ell).$$

In English: A can be covered by a finite number of balls of radius ε.

(d) The set A is <u>cover compact</u> if A satisfies:

if $\mathcal{S} \subseteq \mathcal{T}$ and $A \subseteq \left(\bigcup_{U \in \mathcal{S}} U\right)$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $U_1, \ldots, U_\ell \in \mathcal{S}$ such that

$$A \subseteq U_1 \cup U_2 \cup \cdots \cup U_\ell.$$

In English: Every open cover has a finite subcover.

<u>Theorem</u>

$$\begin{array}{ccccc} \text{cover compact} & \Rightarrow & \text{ball compact} & \Rightarrow & \text{bounded} \\ \Downarrow\Uparrow & \Leftarrow & + & & \\ \text{sequentially compact} & \Rightarrow & \text{Cauchy compact} & \Rightarrow & \text{closed in } X. \end{array}$$

<u>Theorem</u> For $\mathbb{R}^n$ with the standard topology

ball compact $\Leftrightarrow$ bounded

and

Cauchy compact $\Leftrightarrow$ closed in $\mathbb{R}^n$

<u>Proposition</u> Let (X,d) be a complete metric space.
Let $A \subseteq X$.
If A is closed then A is complete.

<u>Proof</u> Assume A is closed in X.

To show: A is complete

To show: If $(a_1, a_2, \ldots)$ is a Cauchy sequence in A then $(a_1, a_2, \ldots)$ converges in A.

Assume $(a_1, a_2, \ldots)$ is a Cauchy sequence in A.

Then $(a_1, a_2, \ldots)$ is a Cauchy sequence in X.

Since X is complete $\lim\limits_{n\to\infty} a_n$ exists in X.

To show: $\lim\limits_{n\to\infty} a_n$ is an element of A.

Since A is closed then

$$A = \bar{A} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists } (x_1, x_2, \ldots) \text{ in } X \\ \text{such that } z = \lim\limits_{n\to\infty} x_n \end{array} \right\}$$

So $\lim\limits_{n\to\infty} a_n \in \bar{A} = A$.

So $(a_1, a_2, \ldots)$ converges in A.

So A is complete. //

<u>Corollary</u> Let $A \subseteq \mathbb{R}$, where $\mathbb{R}$ has the standard topology.
If A is closed then A is complete.

<u>Proposition</u> Let $A \subseteq \mathbb{R}$, where $\mathbb{R}$ has the standard topology.

If A is bounded then A is ball compact.

<u>Proof</u> Assume A is bounded.

To show: A is ball compact.

Since A is bounded there exists $x \in \mathbb{R}$ and $M \in \mathbb{R}_{>0}$ such that $A \subseteq (x-M, x+M)$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, x_2, \ldots x_\ell \in \mathbb{R}$ such that $A \subseteq B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_\ell)$.

Assume $\varepsilon \in \mathbb{R}_{>0}$.

To show: There exists $\ell \in \mathbb{Z}_{>0}$ and $x_1, \ldots, x_\ell \in \mathbb{R}$ such that $A \subseteq B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_\ell)$.

Let $\ell \in \mathbb{Z}_{>0}$ be such that $\ell\frac{\varepsilon}{2} > 2M$.

Let $x_1 = x-M$, $x_2 = x-M+\frac{\varepsilon}{2}$, $\ldots$, $x_\ell = x-M+\ell\frac{\varepsilon}{2}$.

Then

$$\begin{aligned} A \subseteq (x-M, x+M) &\subseteq (x_1-\tfrac{\varepsilon}{2}, x_1+\tfrac{\varepsilon}{2}) \cup (x_2-\tfrac{\varepsilon}{2}, x_2+\tfrac{\varepsilon}{2}) \cup \cdots \cup (x_\ell-\tfrac{\varepsilon}{2}, x_\ell+\tfrac{\varepsilon}{2}) \\ &= B_\varepsilon(x_1) \cup B_\varepsilon(x_2) \cup \cdots \cup B_\varepsilon(x_\ell) \end{aligned}$$

So A is ball compact. //