Theorem Let (X, d_X) be a metric space and let $(Y, \mathcal{T}_Y)$ be a topological space. Let $f: X \to Y$ be a function. Then

f is continuous if and only if

(*) if $x_1, x_2, \ldots$ is a sequence and $\lim_{n\to\infty} x_n$ exists then $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$

Proof $\Rightarrow$ Assume f is continuous.

To show: f satisfies (*).

Assume $x_1, x_2, \ldots$ is a sequence and $\lim_{n\to\infty} x_n = a$.

To show: $f(a) = \lim_{n\to\infty} f(x_n)$.

To show: If $N \in \mathcal{N}(f(a))$ then N contains all but a finite number of $f(x_1), f(x_2), \ldots$

Assume $N \in \mathcal{N}(f(a))$.

Since f is continuous $f^{-1}(N) \in \mathcal{N}(a)$.

So $f^{-1}(N)$ contains all but a finite number of $x_1, x_2, \ldots$

So N contains all but a finite number of $f(x_1), f(x_2), \ldots$

So f satisfies (*).

$\Leftarrow$ To show: If f is not continuous then f does not satisfy $(*)$.

Assume f is not continuous.

Then there exists a such that f is not continuous at a.

So there exists $N \in \mathcal{N}(f(a))$ such that $f^{-1}(N) \notin \mathcal{N}(a)$.

To show: There exists a sequence $x_1, x_2, \ldots$ such that $\lim_{n\to\infty} x_n$ exists and $\lim_{n\to\infty} f(x_n) \neq f(\lim_{n\to\infty} x_n)$.

Since $f^{-1}(N) \notin \mathcal{N}(a)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $f^{-1}(N) \not\supseteq B_{\frac{1}{\ell}}(a)$.

Let $x_1 \in B_{\frac{1}{\ell}}(a) \cap f^{-1}(N)^c$, $x_2 \in B_{\frac{1}{\ell+1}}(a) \cap f^{-1}(N)^c$, ...

To show: (a) $\lim_{n\to\infty} x_n = a$

(b) $\lim_{n\to\infty} f(x_n) \neq f(a)$

(a) To show: If $P \in \mathcal{N}(a)$ then P contains all but a finite number of $x_1, x_2, \ldots$.

Assume $P \in \mathcal{N}(a)$.

To show: P contains all but a finite number of $x_1, x_2, \ldots$

Since $P \in \mathcal{N}(a)$ then there exists $r \in \mathbb{Z}_{>0}$ such that $P \supseteq B_{\frac{1}{r}}(a)$.

Let $k \in \mathbb{Z}_{>0}$ be such that $r = \ell + k$ (let $k = 0$ if $r < \ell$). Then P contains $x_k, x_{k+1}, x_{k+2}, \ldots$

So P contains all but a finite number of $x_1, x_2, \ldots$

So $\lim_{n \to \infty} x_n = a$.

(b) To show: $\lim_{n \to \infty} f(x_n) \neq f(a)$.

To show: There exists $M \in \mathcal{N}(f(a))$ such that ~~M does~~ $M^c \cap \{f(x_1), f(x_2), \ldots\}$ is infinite.

Let $M = N$.

To show: $M^c \cap \{f(x_1), f(x_2), \ldots\}$ is infinite.

To show: $N^c \cap \{f(x_1), f(x_2), \ldots\}$ is infinite.

Since $x_j \in f^{-1}(N)^c$ then

$$f(x_j) \notin N, \quad \text{for } j \in \{1, 2, \ldots\}.$$

So $\{f(x_1), f(x_2), \ldots\} \subseteq N^c$.

So $N^c \cap \{f(x_1), f(x_2), \ldots\} = \{f(x_1), f(x_2), \ldots\}$ is infinite.

So $\lim_{n \to \infty} f(x_n) \neq f(a)$.

So f does not satisfy (*).