Metric and Hilbert spaces, Lecture 14, 26.08.2014
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Theorem Let (X, d) be a metric space and $A \subseteq X$.

$$\bar{A} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists } (a_1, a_2, \ldots) \text{ in } A \\ \text{with } \lim_{n\to\infty} a_n = z \end{array} \right\}$$

Proof Let $R = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists } (a_1, a_2, \ldots) \text{ in } A \\ \text{with } \lim_{n\to\infty} a_n = z \end{array} \right\}$

To show: (a) $R \subseteq \bar{A}$

(b) $\bar{A} \subseteq R$

(a) To show: If $z \in R$ then $z \in \bar{A}$.

Assume $z \in R$.

To show: $z \in \bar{A}$

We know there exists $(a_1, a_2, \ldots)$ in A with $\lim_{n\to\infty} a_n = z$.

To show: z is a close point to A.

To show: If $V \in \mathcal{N}(z)$ then ~~the~~ $V \cap A \neq \emptyset$.

Assume $V \in \mathcal{N}(z)$.

Since $\lim_{n\to\infty} a_n = z$ then V contains all but a finite number of elements of $\{a_1, a_2, \ldots\}$.

So $V \cap A \neq \emptyset$.

So z is a close point to A.

So $z \in \bar{A}$.

(b) To show: $\overline{A} \subseteq R$.

To show: If $z \in \overline{A}$ then $z \in R$

Assume $z \in \overline{A}$.

To show: $z \in R$

To show: There exists a sequence $(a_1, a_2, \ldots)$ in A with $z = \lim_{n\to\infty} a_n$.

Using that z is a close point to A,

let $a_1 \in B_1(z) \cap A$, $a_2 \in B_{\frac{1}{2}}(z) \cap A$, $a_3 \in B_{\frac{1}{3}}(z) \cap A$, ...

To show: $z = \lim_{n\to\infty} a_n$

To show: If $N \in \mathcal{N}(z)$ then N contains all but a finite number of $a_1, a_2, \ldots$.

Let $N \in \mathcal{N}(z)$

Then there exists $\varepsilon \in \mathbb{R}_{>0}$ such that $B_\varepsilon(z) \subseteq N$.

Let $\ell \in \mathbb{Z}_{>0}$ with $\frac{1}{\ell} < \varepsilon$.

Then $a_n \in B_{\frac{1}{n}}(z) \subseteq B_{\frac{1}{\ell}}(z) \subseteq B_\varepsilon(z) \subseteq N$, for $n \in \mathbb{Z}_{\geq \ell}$.

So $a_\ell, a_{\ell+1}, \ldots \in N$.

So N contains all but a finite number of a_j.

So $\lim_{n\to\infty} a_n = z$.

So $z \in R$.

So $\overline{A} \subseteq R$. //.

Hausdorff spaces

<u>Example</u> $X = \{0,1\}$ with $\mathcal{T} = \{\phi, \{0\}, X\}$.

$\lim_{x\to 0} x = 1$, since if $N \in \mathcal{N}(1)$ then there exists $P \in \mathcal{N}(0)$ such that $N \supseteq P$.

$\lim_{x\to 0} x = 0$, since if $N \in \mathcal{N}(0)$ then there exists $P \in \mathcal{N}(0)$ such that $N \supseteq P$.

Similarly, if $(x_1, x_2, \ldots)$ is the sequence $(0, 0, \ldots)$ then $\lim_{n\to\infty} x_n = 1$ and $\lim_{n\to\infty} x_n = 0$.

<u>Proposition</u> Let (Y,d) be a metric space (or a Hausdorff topological space).

(a) Let $(X,\mathcal{T})$ be a topological space and let $f: X \to Y$ be a function. Let $a \in X$.

If $\lim_{x\to a} f(x)$ exists then it is unique.

(b) Let $(y_1, y_2, \ldots)$ be a sequence in Y.

If $\lim_{n\to\infty} y_n$ exists then it is unique.

A <u>Hausdorff topological space</u> is a topological space $(X,\mathcal{T})$ such that

if $x, y \in X$ and $x \neq y$ then there exist $U, V \in \mathcal{T}$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$.