Metric and Hilbert spaces, Lecture 13, 25.08.2015 ①

Limits

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

Let $f: X \to Y$ be a function. Let $a \in X$ and $y \in Y$.

Write

$$\lim_{x \to a} f(x) = y \qquad \text{if } f \text{ satisfies}$$

if $N \in \mathcal{N}(y)$ then there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

Let $(x_1, x_2, \ldots)$ be a sequence in X. Let $y \in Y$.

Write

$$\lim_{n \to \infty} x_n = y \qquad \text{if } (x_1, x_2, \ldots) \text{ satisfies}$$

if $N \in \mathcal{N}(y)$ then N contains all but a finite number of elements of $\{x_1, x_2, \ldots\}$.

In other words:

if $N \in \mathcal{N}(y)$ then there exists $\ell \in \mathbb{Z}_{>0}$ such that $N \supseteq \{x_\ell, x_{\ell+1}, \ldots\}$.

For metric spaces:

$\lim_{x\to a} f(x) = y$ if f satisfies

if $B_\varepsilon(y)$ is an open ball at y then

there exists $B_\delta(a)$, an open ball at x,

such that $B_\varepsilon(y) \supseteq f(B_\delta(a))$.

i.e.

$\lim_{x\to a} f(x) = y$ if f satisfies

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such

that if $x \in X$ and $d_X(x,a) < \delta$ then $d_Y(f(x),y) < \varepsilon$

For metric spaces:

$\lim_{n\to\infty} x_n = y$ if $(x_1, x_2, \ldots)$ satisfies

if $B_\varepsilon(y)$ is an open ball at y then

there exists $\ell \in \mathbb{Z}_{>0}$ such that $B_\varepsilon(y) \supseteq \{x_{\ell+1}, \ldots\}$.

i.e. $\lim_{n\to\infty} x_n = y$ if $(x_1, x_2, \ldots)$ satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, y) < \varepsilon$.

Proposition

(a) Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Let $f: X \to Y$ be a function and let $a \in X$. ~~Write~~ then

f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.

(b) Let (X, d_X) and (Y, d_Y) be metric spaces. Let $f: X \to Y$ be a function. Let $a \in X$ and $y \in Y$.

Then $\lim_{x \to a} f(x) = y$ if and only if f satisfies

(*) if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $\delta \in \mathbb{R}_{>0}$ such that if $x \in X$ and $d(x, a) < \delta$ then $d(f(x), y) < \varepsilon$.

(c) Let (X, d_X) be a metric space. Let $A \subseteq X$.

Then

$$\bar{A} = \left\{ z \in X \;\middle|\; \begin{array}{l} \text{there exists a sequence } (a_1, a_2, \ldots) \text{ in } A \\ \text{such that } z = \lim_{n \to \infty} a_n \end{array} \right\}$$

(d) Let (X, d_X) be a metric space and let $(Y, \mathcal{T}_Y)$ be a topological space. Let $f: X \to Y$ be a function.

Then

f is continuous if and only if f satisfies

if $(x_1, x_2, \ldots)$ is a sequence in X and $\lim_{n \to \infty} x_n$ exists then $\lim_{n \to \infty} f(x_n) = f\left(\lim_{n \to \infty} x_n\right)$.

First countable, second countable and separable

Let $(X, \mathcal{T}_X)$ be a topological space.

$(X, \mathcal{T}_X)$ is <u>first countable</u> if ~~then~~ it satisfies:

if $a \in X$ then $\mathcal{N}(a)$ is countably generated.

$(X, \mathcal{T}_X)$ is <u>second countable</u>

if $\mathcal{T}_X$ is countably generated.

$(X, \mathcal{T}_X)$ is <u>separable</u> if

X has a countable dense subset.

More precisely:

$(X, \mathcal{T}_X)$ is <u>first countable</u> if it satisfies:

if $a \in X$ then there exist $N_1, N_2, \ldots \in \mathcal{N}(a)$

such that if $N \in \mathcal{N}(a)$ then there exists

$j \in \mathbb{Z}_{>0}$ such that $N \supseteq N_j$.

$(X, \mathcal{T}_X)$ is <u>second countable</u> if

there exist $U_1, U_2, \ldots \in \mathcal{T}_X$ such that

$\mathcal{T}_X = \{\text{unions of } U_j\} = \{U_{i_1} \cup U_{i_2} \cup \cdots \mid i_1, i_2 \ldots \in \mathbb{Z}_{>0}\}$

$(X, \mathcal{T}_X)$ is <u>separable</u> if there exist

$x_1, x_2, \ldots \in X$ such that $\overline{\{x_1, x_2, \ldots\}} = X$.

$$\begin{array}{ccccc} \text{first countable} & \Leftarrow & \text{second countable} & \Rightarrow & \text{separable} \\ & \not\Rightarrow & & \not\Leftarrow & \\ & \text{$\mathbb{R}$ with the discrete topology} & & \text{$\mathbb{R}$ with the topology $\mathcal{T} = \{\text{unions of } [a,b)\}$} & \end{array}$$

<u>HW</u>: Show that $\mathbb{R}$ with the topology

$$\mathcal{T} = \{ U \subseteq \mathbb{R} \mid U^c \text{ is finite} \}$$

is not first countable.