Metric and Hilbert Lecture 12, 20 August 2015 ①

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Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A function $f: X \to Y$ is <u>continuous</u> if f satisfies

(C) if $V \in \mathcal{T}_Y$ then $f^{-1}(V) \in \mathcal{T}_X$.

Let $a \in X$. A function $f: X \to Y$ is <u>continuous at a</u> if f satisfies

(Cat) if $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.

<u>Proposition</u> Let X and Y be topological spaces. A function $f: X \to Y$ is continuous if and only if f satisfies

if $a \in X$ then f is continuous at a.

<u>Proof</u>: $\Rightarrow$: Assume $f: X \to Y$ is continuous

To show: If $a \in X$ then f is continuous at a.

Assume $a \in X$.

To show: If $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$

Assume $N \in \mathcal{N}(f(a))$.

Then there exists $V \in \mathcal{T}_Y$ with $f(a) \in V \subseteq N$.

So $a \in f^{-1}(V) \subseteq f^{-1}(N)$.

Since f is continuous $f^{-1}(V)$ is open in X.

So $f^{-1}(N) \in \mathcal{N}(a)$.

So f is continuous at a.

$\Leftarrow$: Assume that f satisfies

if $a \in X$ then f is continuous at a.

To show: f is continuous.

To show: If V is open in Y then $f^{-1}(V)$ is open in X.

Assume V is open in Y.

To show: $f^{-1}(V)$ is open in X.

To show: If $a \in f^{-1}(V)$ then a is an interior point of $f^{-1}(V)$.

Assume $a \in f^{-1}(V)$

To show: a is an interior point of $f^{-1}(V)$

Since $f(a) \in V$ and V is open in Y then $V \in \mathcal{N}(f(a))$.

Since f is continuous at a then $f^{-1}(V) \in \mathcal{N}(a)$.

So there exists U open in X with $a \in U \subseteq f^{-1}(V)$

So a is an interior point of $f^{-1}(V)$.

So $f^{-1}(V)$ is open in X.

So f is continuous. //

Let $(Y, \mathcal{T}_Y)$ be a topological space.
Let $\mathcal{G}$ be a filter on Y.

A <u>limit point</u> of $\mathcal{G}$ is an element

$y \in Y$ such that $\mathcal{G} \supseteq \mathcal{N}(y)$.

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
Let $f: X \to Y$ be a function and let $a \in X$. Write

$$y = \lim_{x \to a} f(x) \quad \text{if} \quad \mathcal{F}(f(\mathcal{N}(a))) \supseteq \mathcal{N}(y),$$

where $\mathcal{F}(f(\mathcal{N}(a)))$ is the filter generated by

$$f(\mathcal{N}(a)) = \{ f(N) \mid N \in \mathcal{N}(a) \}.$$

<u>Proposition</u> Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.
Let $f: X \to Y$ be a function and let $a \in X$. Then
f is continuous at a if and only if $f(a) = \lim_{x \to a} f(a)$.

<u>Proof:</u> $\Rightarrow$ Assume f is continuous at a.

To show: $f(a) = \lim_{x \to a} f(x)$

To show: $\mathcal{F}(f(\mathcal{N}(a))) \supseteq \mathcal{N}(f(a))$

To show: If $N \in \mathcal{N}(f(a))$ then $N \in \mathcal{F}(f(\mathcal{N}(f(a)))$

Assume $N \in \mathcal{N}(f(a))$.

To show: $N \in \mathcal{F}(f(\mathcal{N}(a)))$.

To show: N contains a set in $f(\mathcal{N}(a))$.

To show: There exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$.

Since $N \in \mathcal{N}(f(a))$ and f is continuous at a then

$f^{-1}(N) \in \mathcal{N}(a)$.

Let $P = f^{-1}(N)$.

To show: $N \supseteq f(P)$.

$N \supseteq \cancel{N \supseteq} f(f^{-1}(N)) = f(P)$.

So $\mathcal{F}(f(\mathcal{N}(a))) \supseteq \mathcal{N}(f(a))$.

So $f(a) = \lim_{x \to a} f(x)$.

$\Leftarrow$ Assume $f(a) = \lim_{x \to a} f(a)$.

To show: f is continuous at a.

To show: ~~If~~ $N \in \mathcal{N}(f(a))$ then $f^{-1}(N) \in \mathcal{N}(a)$.

Assume $N \in \mathcal{N}(f(a))$.

To show: $f^{-1}(N) \in \mathcal{N}(a)$.

Since $f(a) = \lim_{x \to a} f(a)$ then $\mathcal{F}(f(\mathcal{N}(a))) \supseteq \mathcal{N}(f(a))$

and $N \in \mathcal{F}(f(\mathcal{N}(a)))$.

So N contains a set in $f(\mathcal{N}(a))$.

So there exists $P \in \mathcal{N}(a)$ such that $N \supseteq f(P)$

~~So at P and~~ Since $f^{-1}(N) \supseteq f^{-1}(f(P)) \supseteq P$, then $f^{-1}(N) \in \mathcal{N}(a)$.

So f is continuous at all