Metric and Hilbert, Lecture 11, 19 August 2015
Univ. of Melbourne

Connected components

Let $(X, \mathcal{T})$ be a topological space. Let $E \subseteq X$.

The set E is connected if there do not exist open sets A and B in X such that

$$A \cap E \neq \emptyset, \quad B \cap E \neq \emptyset, \quad E \subseteq A \cup B, \text{ and}$$
$$(A \cap E) \cap (B \cap E) = \emptyset.$$

Define an equivalence relation on X by

$x \sim y$ if there exists a connected $E \subseteq X$ with $x \in E$ and $y \in E$.

HW: Show that $\sim$ is an equivalence relation

The connected components of X are the equivalence classes of $\sim$.

HW: Let $x \in X$. Show that the connected component of X which contains x is equal to

$$C_x = \left(\bigcup_{\substack{E \subseteq X \\ E \text{ connected} \\ x \in E}} E \right)$$

Let $(X, \mathcal{T})$ be a topological space. Let $E \subseteq X$.
The set E is path connected if E satisfies:

if $p, q \in E$ then there exists a continuous function $f : [0,1] \to E$ with $f(0) = p$ and $f(1) = q$.

HW: Show that if E is path connected then E is connected.

HW: Give an example of a set E which is connected but not path connected.

connected $\overset{\Leftarrow}{\not\Rightarrow}$ path connected.

Important theorem Let $X = \mathbb{R}$ with the standard topology. Let $E \subseteq X$.

(a) E is connected if and only if E is an interval.

(b) E is compact if and only if E is closed and bounded.

Complete metric spaces

Let (X,d) be a metric space.

A **sequence** in X is a function $f: \mathbb{Z}_{>0} \to X$

Let $(x_1, x_2, \ldots)$ be a sequence in X.

The sequence $(x_1, x_2, \ldots)$ **converges** if $x = \lim_{n\to\infty} x_n$ exists,

i.e. there exists $x \in X$ such that

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $n \in \mathbb{Z}_{\geq N}$ then $d(x_n, x) < \varepsilon$.

Write $\lim_{n\to\infty} x_n = x$ if $(x_1, x_2, \ldots)$ converges to x.

The sequence $(x_1, x_2, \ldots)$ is **Cauchy** if it satisfies:

if $\varepsilon \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that

if $r, s \in \mathbb{Z}_{\geq N}$ then $d(x_r, x_s) < \varepsilon$.

HW: Show that if $(x_1, x_2, \ldots)$ converges then $(x_1, x_2, \ldots)$ is Cauchy.

HW: Give an example of a sequence $(x_1, x_2, \ldots)$ that is Cauchy but does not converge.

converges $\underset{\not\Leftarrow}{\Rightarrow}$ Cauchy

A <u>complete metric space</u> is a metric space (X,d) such that

if $(x_1, x_2, \ldots)$ is a sequence in X and
$(x_1, x_2, \ldots)$ is Cauchy
then $(x_1, x_2, \ldots)$ converges.

<u>Important theorem</u> Let $X = \mathbb{R}$ with the standard metric. Then X is complete.

<u>HW</u> Give an example of a metric space that is not complete.

<u>Bounded continuous functions</u>

Let (X, d_X) and (Y, d_Y) be metric spaces.
Let

$$C_b(X,Y) = \left\{ f: X \to Y \;\middle|\; \begin{matrix} f \text{ is continuous} \\ \text{and } f(X) \text{ is bounded} \end{matrix} \right\}$$

with $d_\infty : C_b(X,Y) \times C_b(X,Y) \to \mathbb{R}_{\geq 0}$ given by

$$d_\infty(f,g) = \sup\{ d_Y(f(x), g(x)) \mid x \in X \}.$$

<u>Theorem</u> If Y is a complete metric space then $C_b(X,Y)$ is a complete metric space.