Metric and Hilbert spaces, Lecture 10, 18 August 2015
Univ. of Melbourne

Theorem Let (X,d) be a complete metric space.

Let $U_1, U_2, \ldots$ be open dense subsets of X.

Then
$$\bigcap_{n \in \mathbb{Z}_{>0}} U_n = U_1 \cap U_2 \cap \cdots \quad \text{is dense in } X.$$

Proof: To show: $\overline{\left(\bigcup_{n \in \mathbb{Z}_{>0}} U_n\right)} = X$.

To show: If $x \in X$ then x is a close point to $\bigcap_{n \in \mathbb{Z}_{>0}} U_n$.

Assume $x \in X$.

To show: x is a close point to $\bigcap_{n \in \mathbb{Z}_{>0}} U_n$.

To show: If $\varepsilon \in \mathbb{R}_{>0}$ then $B_\varepsilon(x) \cap \left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n\right) \neq \emptyset$.

Assume $\varepsilon \in \mathbb{R}_{>0}$

To show: There exists $y \in B_\varepsilon(x) \cap \left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n\right)$

Let $x_0 = x$ and $\varepsilon_0 = \varepsilon$. Using that U_n is open and dense let $x_k \in X$ and $\varepsilon_{k+1} \in \mathbb{R}_{>0}$ be such that
$$B_{3\varepsilon_{k+1}}(x_{k+1}) \subseteq B_{\varepsilon_k}(x_k) \cap U_{k+1} \text{ and } \varepsilon_{k+1} < \frac{\varepsilon_k}{3}$$
and let
$$y = \lim_{n \to \infty} x_n.$$

To show: (a) y exists

(b) $y \in B_\varepsilon(x) \cap \left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n \right)$

(a) To show: $y = \lim_{n \to \infty} x_n$ exists

To show: $(x_1, x_2, \ldots)$ converges.

Using that X is complete,

to show: $(x_1, x_2, \ldots)$ is Cauchy

To show: If $\delta \in \mathbb{R}_{>0}$ then there exists $N \in \mathbb{Z}_{>0}$ such that if $r, s \in \mathbb{Z}_{\geq N}$ then $d(x_r, x_s) < \delta$.

Assume $\delta \in \mathbb{R}_{>0}$

Let $N \in \mathbb{Z}_{>0}$ be such that

$$\frac{1}{3^N} \cdot \frac{3}{2} \cdot \varepsilon < \delta$$

To show: If $r, s \in \mathbb{Z}_{\geq N}$ then $d(x_r, x_s) < \delta$.

Assume $r, s \in \mathbb{Z}_{\geq N}$.

To show: $d(x_r, x_s) < \delta$.

Using the triangle inequality,

$$d(x_r, x_s) \leq d(x_r, x_{r+1}) + d(x_{r+1}, x_{r+2}) + \cdots + d(x_{s-1}, x_s)$$

$$< \varepsilon_r + \varepsilon_{r+1} + \cdots + \varepsilon_{s-1}$$

Since $\varepsilon_{k+1} < \frac{\varepsilon_k}{3}$ for $k \in \mathbb{Z}_{>0}$, then $\varepsilon_{k+l} < \frac{1}{3^l}\varepsilon_k$ for $l \in \mathbb{Z}_{>0}$ and

$$d(x_r, x_s) \le \varepsilon_r + \varepsilon_{r+1} + \cdots + \varepsilon_{s-1}$$

$$< \left(\frac{1}{3^r} + \frac{1}{3^{r+1}} + \cdots + \frac{1}{3^{s-1}}\right)\varepsilon < \frac{1}{3^r}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots\right)\varepsilon$$

$$= \frac{1}{3^r}\left(\frac{1}{1-\frac{1}{3}}\right)\varepsilon = \frac{1}{3^r}\cdot\frac{3}{2}\cdot\varepsilon \le \frac{1}{3^N}\cdot\frac{3}{2}\cdot\varepsilon < \delta$$

So $(x_1, x_2, \ldots)$ is Cauchy.

So $(x_1, x_2, \ldots)$ converges.

So $y = \lim_{n\to\infty} x_n$ exists.

(b) To show: $y \in B_\varepsilon(x) \cap \left(\bigcap_{n \in \mathbb{Z}_{>0}} U_n\right)$

Using that

$$B_{3\varepsilon_k}(x_k) \subseteq B_{\varepsilon_{k-1}}(x_{k-1}) \cap U_k \subseteq U_k, \text{ it suffices}$$

to show: If $k \in \mathbb{Z}_{>0}$ then $d(y, x_k) < 3\varepsilon_k$.

Assume $k \in \mathbb{Z}_{>0}$

To show: $d(y, x_k) < 3\varepsilon_k$

$$d(y, x_k) = d\left(\lim_{n\to\infty} x_n, x_k\right)$$

$$= \lim_{n\to\infty} d(x_n, x_k), \quad \text{since } d(\cdot, x_k): X \to \mathbb{R}_{\ge 0} \text{ is continuous,}$$

$$\le \lim_{n\to\infty}\left(d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+2}) + \cdots + d(x_{n-1}, x_n)\right)$$

$$\le \lim_{n\to\infty}\left(\varepsilon_k + \frac{1}{3}\varepsilon_k + \frac{1}{3^2}\varepsilon_k + \cdots + \frac{1}{3^{n-k}}\varepsilon_k\right)$$

$$= \varepsilon_k \lim_{n\to\infty}\left(1 + \frac{1}{3} + \frac{1}{3^2} + \cdots + \frac{1}{3^{n-k}}\right) = \varepsilon_k\left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right)$$

$$= \varepsilon_k \cdot \frac{1}{1-\frac{1}{3}} = \varepsilon_k \frac{3}{2} < 3\varepsilon_k.$$

So $y \in B_\varepsilon(x) \cap \left(\bigcup_{n\in\mathbb{Z}_{>0}} U_n\right)$

So $B_\varepsilon(x) \cap \left(\bigcup_{n\in\mathbb{Z}_{>0}} U_n\right) \neq \emptyset$.

So x is a close point to $\bigcup_{n\in\mathbb{Z}_{>0}} U_n$.

So $\overline{\left(\bigcup_{n\in\mathbb{Z}_{>0}} U_n\right)} = X$.

So $\left(\bigcup_{n\in\mathbb{Z}_{>0}} U_n\right)$ is dense in X.